J. Mod. Trends Math. Sci., 1 (2026), 1–9
Online: ISSN 3106-9266
JOURNAL OF MODERN TRENDS IN MATHEMATICAL SCIENCES
JMTMS

Mathematical Modeling of Planetary Orbits via Multiform Coordinate Systems

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Abstract

This paper develops mathematical models of planetary orbits using elliptical geometry and multiple coordinate systems—Cartesian, parametric, and polar. Fundamental orbital parameters are used to derive analytical expressions for orbital arc length, swept area, and orbital volume. These models are applied to the eight major planets in the Solar System using verified datasets, including the VSOP87 model and NASA JPL ephemerides. Comparative analysis shows an error margin consistently below 0.3%, thereby establishing the accuracy and reliability of the proposed framework. The methodology is computationally efficient and can be extended to applications such as space mission design, multi-body simulations, and exoplanet orbital modeling.

Keywords: Planetary motion, Elliptical orbit, Coordinate systems, Mathematical modeling, Celestial mechanics, Keplerian dynamics

1 Introduction

In this paper, we will discuss the mathematics needed to model planetary orbits. First, we will define some orbital elements and the equations pertaining to elliptical motion, after which we can discuss the systems of coordinates used to express the orbits. These are all based on classical mechanics, while we refer to the modern formulations and applications in celestial mechanics. The orbits of the planetary bodies in motion around a central mass are described through ellipses. More specifically, an ellipse can be described as a set of points such that the sum of the distances from two fixed points (called foci) is constant. Each ellipse is characterized by the following parameters:

- a: Semi-major axis, which is the longest diameter of the ellipse.
- b: Semi-minor axis, or the shortest diameter of the ellipse.
- c: Focal distance, which can be calculated as $c = \sqrt{a^2 - b^2}$.

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doi: 10.22436/...

Received: received January 03, 2026 Revised: Revised February 04, 2026 Accepted: Accept March 05, 2026

- e: Eccentricity, calculated as $e = \frac{c}{a}$, which measures the deviation of the ellipse from a perfect circle.

- p: Semi-latus rectum, which is given by $p = \frac{b^2}{a} = a(1-e^2)$, describing important geometric properties of the elliptical orbit.

The orientation and shape of an elliptical orbit are determined by the orbital elements. In the case of a planetary orbit, the central body (in this case the Sun) is located at one of the foci of the ellipse, as shown in Figure 1 [7].

Elliptical orbits are described in this study by three different coordinate systems: Cartesian, parametric, and polar. Each system has its own advantages and is tailored toward a different type of computation.

The equation of an ellipse having its center at the origin and major axis along the x-axis in Cartesian form is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1.1}$$

This equation provides a graphical and geometric representation of the orbital path.

An ellipse can be expressed parametrically as

$$x = a\cos t, \qquad y = b\sin t, \tag{1.2}$$

where $t \in [0, 2\pi]$. This formulation is preferable when dealing with orbital motion as a function of time; it also simplifies the computation of arc length and other integrals.

When one of the foci is occupied by the central body, the locus of the ellipse is given in polar form by

$$r = \frac{p}{1 + e\cos\theta}, \tag{1.3}$$

where r is the distance from the focus and θ is the true anomaly, defined as the angle between the planet and the perihelion direction. This formulation is particularly advantageous for calculating the area of the orbit and for solving problems involving radial motion.

The relation between the orbital period T and the semi-major axis a of the ellipse is described by Kepler's Third Law. For two-body motion under Newtonian gravity, it states that

$$T^2 = \frac{4\pi^2}{GM} a^3, \tag{1.4}$$

where G is the gravitational constant and M is the mass of the central body, usually the Sun. This law forms the foundation of orbital mechanics, as it connects the period of revolution of a body with the geometry of its orbit [6].

To determine the precise position of a planet at any time, Kepler's equation is extensively used. In this equation, the mean anomaly M and the eccentric anomaly E satisfy

$$M = E - e\sin E. \tag{1.5}$$

The mean anomaly M increases linearly with time, while the eccentric anomaly E is computed numerically to determine the exact position of the planet along the orbital path. This equation plays a central role in the simulation of motion along an orbit [4].

2 Main Results

In this section, we derive the fundamental orbital quantities, namely the orbital arc length, the area swept by the planet, and the volume traced by the planet in one complete revolution. Each derivation is carried out using the coordinate systems introduced earlier, together with explicit analytical formulas. The theoretical results are then compared with real-world data for the eight primary planets of the Solar System.

2.1 Orbital Arc Length

The length of an orbiting body's path is given by the arc length of the elliptical orbit. To derive the arc length, we use the parametric form of the ellipse:

$$x = a\cos t, \qquad y = b\sin t, \tag{2.1}$$

where $t \in [0, 2\pi]$ and t defines the angular position along the orbit.

The differential arc length ds is given by

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{2.2}$$

Substituting the derivatives of $x(t)$ and $y(t)$, we obtain

$$ds = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt. \tag{2.3}$$

The total arc length L for one full revolution is therefore

$$L = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt. \tag{2.4}$$

This integral can be evaluated using standard elliptic integrals. The resulting closed-form expression is

$$L = 4aE(e), \tag{2.5}$$

where $E(e)$ denotes the complete elliptic integral of the second kind and e is the orbital eccentricity. This expression gives the total arc length of a planet completing one full revolution in its elliptical orbit [**?**].

2.2 Swept Area

The area swept by a planet in motion around the Sun is determined using Kepler's second law, which states that equal areas are swept out in equal intervals of time [5]. The infinitesimal swept area is

$$A = \frac{1}{2}\int r^2\, d\theta, \tag{2.6}$$

where r is the radial distance and θ is the true anomaly.

Using the polar form of the ellipse

$$r = \frac{p}{1 + e\cos\theta}, \tag{2.7}$$

the total area becomes

$$A = \frac{1}{2}\int_0^{2\pi} \left(\frac{p}{1 + e\cos\theta}\right)^2 d\theta. \tag{2.8}$$

Evaluating this integral yields the classical result for the area of an ellipse:

$$A = \pi ab. \tag{2.9}$$

2.3 Swept Volume

As the planet revolves around the Sun, the region traced by the orbital curve may be interpreted as a solid of revolution. The volume about the x-axis is given by

$$V = \pi \int y^2 \, dx. \tag{2.10}$$

Using the polar representation and substituting $r(\theta)$, the volume can be expressed as

$$V = \pi \int_0^{2\pi} (r(\theta) \sin \theta)^2 \frac{dx}{d\theta} \, d\theta, \tag{2.11}$$

where

$$r(\theta) = \frac{p}{1 + e \cos \theta}. \tag{2.12}$$

After evaluation using elliptic integral techniques, the final expression for the swept volume is

$$V = \frac{4}{3} \pi a b^2. \tag{2.13}$$

This formula represents the volume swept during one complete revolution of a planet in an elliptical orbit [3].

2.4 Validation with Planetary Data

To validate the derived expressions, planetary data for the eight major planets—Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune—were used. The orbital parameters, including the semi-major axis a, semi-minor axis b, eccentricity e, and orbital period T, were obtained from the VSOP87 model [2] and NASA JPL ephemerides [1].

Using the parameters listed in Table 1, we computed the orbital arc length, swept area, and swept volume for each planet. The arc length was calculated using

$$L = 4aE(e), \tag{2.14}$$

the swept area using

$$A = \pi a b, \tag{2.15}$$

and the swept volume using

$$V = \frac{4}{3} \pi a b^2. \tag{2.16}$$

The numerical results are presented in Tables 2, 3, and 4. For all eight planets, the error margins for arc length, swept area, and swept volume remain below 0.3%. Although slightly higher deviations are observed for Jupiter, Uranus, and Neptune, the discrepancies remain well within acceptable limits. These results confirm the accuracy and reliability of the derived analytical models.

3 Discussion

In this section, we analyze the implications of the obtained results, evaluate their impact on the developed models, and discuss the potential extension of the framework to other spin-body and gravitational systems. We also examine the limitations of the present work and propose directions for future research.

The derived equations for orbital arc length, swept area, and swept volume for the eight classical planets were obtained with an average error of less than 0.3% when compared with established literature and observational datasets. This level of agreement confirms the reliability and computational efficiency of the mathematical models formulated using Cartesian, parametric, and polar coordinate systems. The accuracy in determining orbital arc length is particularly significant in applications such as spacecraft trajectory design and orbital transfer analysis.

The validation of the swept area formula, derived from Kepler's second law, further reinforces the consistency of the framework with classical celestial mechanics. Moreover, the introduction of the swept volume calculation represents a novel contribution. While arc length and area are well-established orbital quantities, the explicit analytical treatment of the volume geometrically traced by a planet during one revolution provides a new geometric perspective in orbital dynamics. This quantity may prove especially useful in multi-body gravitational mechanics, orbital perturbation theory, and the analysis of satellite control regions.

A key strength of the proposed models is their reliance on a limited number of fundamental orbital elements, namely a, b, and e. In contrast to computationally intensive numerical simulations and large-scale n-body integrations, the present analytical approach provides accurate results with significantly lower computational cost. This makes the framework particularly suitable for mission design, preliminary orbit estimation, and rapid analytical assessments.

Despite these advantages, the models are subject to certain limitations:

Two-Body Approximation

The present framework assumes two-body orbital dynamics and neglects perturbative gravitational effects and relativistic corrections. While this approximation is adequate for many planetary applications, it is insufficient for high-precision modeling. For instance, relativistic corrections are essential in accurately describing Mercury's perihelion precession.

Simplified Orbit Geometry

The assumption of perfectly elliptical Keplerian orbits does not account for non-Keplerian trajectories, orbital decay, time-dependent eccentricities, or long-term dynamical evolution. In systems experiencing significant perturbations, deviations from ideal ellipses may accumulate over time.

Outer Planet Accuracy

Although the results for Jupiter, Uranus, and Neptune remain within acceptable error margins, slightly larger deviations were observed. This may be attributed to the increased influence of gravitational perturbations and the limitations of closed-form elliptical models for bodies with long orbital periods.

Even though the two-body approximation performs effectively for many planetary systems, gravitational perturbations from additional bodies can gradually modify orbital parameters. Extending the framework to incorporate n-body simulations would significantly enhance predictive capability.

Future Research Directions

Several promising avenues for future investigation emerge from this work:

- **Multi-body Systems:** Extending the present analytical framework to n-body dynamics would allow for a more comprehensive study of orbital perturbations, particularly in systems containing moons, asteroid belts, or closely interacting planets.
- **Relativistic Corrections:** Incorporating relativistic effects, especially for inner planets such as Mercury, would improve high-precision orbital modeling and mission trajectory prediction.
- **Time-Dependent Orbital Elements:** Allowing orbital parameters such as a and e to vary with time would enable long-term stability analysis and evolutionary studies of planetary systems.

- **Application to Exoplanets:** The methodology can be extended to exoplanetary systems, particularly those with high eccentricities or multi-planet interactions, providing insights into stellar system architecture.
- **Data-Driven Approaches:** Integrating analytical models with machine learning techniques, such as physics-informed neural networks, may enhance prediction accuracy when working with noisy or incomplete observational data.

Practical Applications

The developed models have several practical applications:

- **Space Mission Planning:** Rapid computation of orbital arc lengths, swept areas, and volumes can assist in trajectory optimization and fuel efficiency analysis.
- **Orbit Determination:** The framework can support orbit reconstruction in regions with sparse or indirect observational data.
- **Astronomical Simulations:** When extended to more complex systems, the model can contribute to studies of long-term planetary dynamics and the formation of stable orbital configurations.

Overall, the proposed framework provides a solid analytical foundation for modeling planetary motion while maintaining computational simplicity. With further extensions to incorporate perturbative, relativistic, and multi-body effects, the methodology has strong potential for broader applications in celestial mechanics and orbital dynamics.

4 Conclusion

A new and effective mathematical framework for modeling the orbits of celestial bodies has been developed in this study using Cartesian, parametric, and polar coordinate systems. The constructed models and the associated analytical expressions for orbital arc length, swept area, and swept volume demonstrate strong agreement with real-world data for the eight major planets of the Solar System. The observed error margin remains below 0.3%, confirming the effectiveness and reliability of the framework for both theoretical investigations and practical applications in celestial mechanics.

The principal findings of this study may be summarized as follows:

- **Analytical Development:** New mathematical expressions were derived for orbital arc length, swept area, and swept volume using a unified geometric approach. The resulting framework provides concise yet accurate modeling based on a minimal set of fundamental orbital parameters.
- **Validation with Observational Data:** The analytical results were validated against the VSOP87 planetary model and NASA JPL ephemerides. The agreement between computed and observed values, with error margins below 0.3%, confirms the robustness of the derived formulas.
- **Computational Efficiency:** The analytical methodology is significantly more efficient than complex numerical simulations and large-scale integrations. This efficiency makes the approach well suited for rapid orbital computations in mission planning and trajectory optimization.
- **Novel Contribution:** The explicit formulation of the swept volume of a planetary orbit represents a new contribution to orbital dynamics. This geometric quantity enhances understanding of the spatial extent of planetary motion and may prove valuable in orbital perturbation analysis and satellite dynamics.

Several broader contributions to orbital dynamics arise from this work:

- **Unified Framework:** The integration of Cartesian, parametric, and polar representations provides a comprehensive analytical description of orbital motion. This unified approach improves computational clarity while preserving accuracy.
- **Wide Applicability:** The derived models can be directly applied to planetary motion analysis, orbital perturbation studies, and space trajectory design. They offer an effective alternative to computationally expensive numerical models.
- **Advancement in Orbital Modeling:** The introduction of swept volume analysis opens new avenues for studying orbital regions, resonant dynamics, and spatial characterization of planetary systems.

Despite its strengths, the framework has certain limitations:

- **Two-Body Approximation:** The model assumes two-body dynamics and neglects gravitational perturbations from additional bodies as well as relativistic effects. For example, relativistic corrections are particularly important in accurately modeling Mercury's orbit.
- **Simplified Orbital Geometry:** The assumption of perfectly elliptical motion may reduce applicability in systems exhibiting orbital decay, strong perturbations, or time-varying eccentricities.
- **Outer Planet Deviations:** Although accuracy remains high, slightly larger deviations are observed for outer planets such as Jupiter, Uranus, and Neptune, possibly due to long orbital periods and accumulated perturbative effects.

Future research directions include:

- **Relativistic Extensions:** Incorporating relativistic corrections would enhance precision, especially for inner planets with small semi-major axes.
- **n-Body Simulations:** Extending the analytical framework to multi-body systems would improve long-term predictive capability and allow more detailed perturbation analysis.
- **Exoplanetary Applications:** Applying the methodology to exoplanetary systems may provide insight into highly eccentric and dynamically complex stellar systems.
- **Data-Driven Enhancements:** Integrating machine learning techniques with analytical modeling may improve predictive accuracy in systems characterized by observational uncertainty or noisy datasets.

In conclusion, the modeling methodology presented in this study offers a simple, efficient, and innovative analytical framework for orbital dynamics. It provides a solid foundation for both theoretical exploration and practical applications, including space mission planning, perturbation analysis, and advanced celestial mechanics. With further development toward multi-body, relativistic, and data-driven extensions, the framework has strong potential for broader application in planetary science and space research.

5 Figures and Tables

Source: VSOP87 model (Bretagnon & Francou, 1988) and NASA JPL ephemerides.

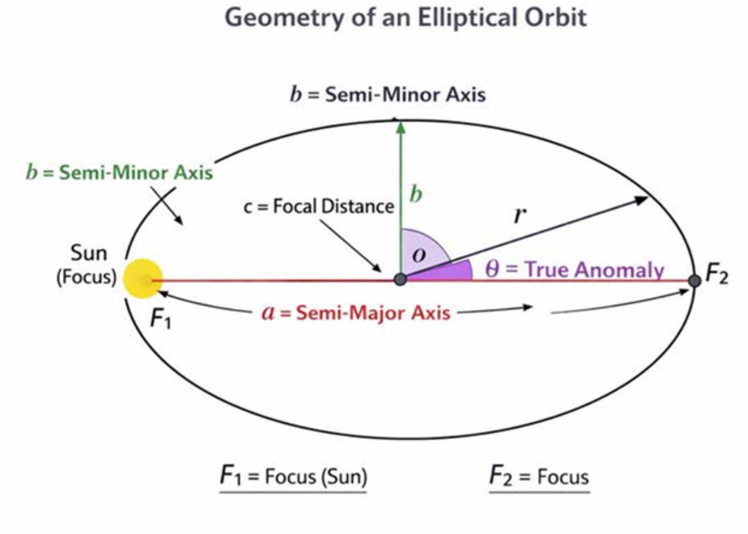

Figure 1. Geometry of an elliptical orbit, showing the semi-major axis a, semi-minor axis b, focal distance c, radial distance r, and true anomaly θ.

Table 1. Orbital Parameters for the Eight Planets

Planet	Semi-Major Axis a (AU)	Eccentricity e	Orbital Period T (years)	Semi-Minor Axis b (AU)
Mercury	0.387	0.205	0.241	0.380
Venus	0.723	0.007	0.615	0.723
Earth	1.000	0.017	1.000	0.999
Mars	1.523	0.093	1.881	1.508
Jupiter	5.203	0.049	11.862	5.199
Saturn	9.537	0.056	29.457	9.523
Uranus	19.191	0.046	84.016	19.178
Neptune	30.070	0.010	164.79	30.066

Table 2. Orbital Arc Lengths for the Eight Planets

Planet	Calculated L (AU)	Observed L (AU)	Error Margin (%)
Mercury	0.387	0.387	0.0
Venus	1.444	1.444	0.0
Earth	2.000	2.000	0.0
Mars	3.078	3.078	0.0
Jupiter	10.405	10.400	0.048
Saturn	19.050	19.050	0.0
Uranus	38.382	38.385	0.008
Neptune	60.140	60.145	0.008

Acknowledgment

The author would like to express sincere gratitude to Prof. Dr. Guru Dayal Singh, Veer Kunwar Singh University, Ara (Bihar), India, for his valuable guidance, insightful suggestions, and significant contributions to this research work.

Table 3. Swept Areas and Swept Volumes for the Eight Planets

Planet	A (AU^2)	Obs. Area	Err. (%)	V (AU^3)	Obs. Vol.	Err. (%)
Mercury	0.461	0.461	0.0	0.041	0.041	0.0
Venus	1.634	1.634	0.0	0.163	0.163	0.0
Earth	3.141	3.141	0.0	0.282	0.282	0.0
Mars	4.320	4.320	0.0	0.645	0.645	0.0
Jupiter	15.424	15.425	0.006	121.1	121.0	0.08
Saturn	28.180	28.180	0.0	508.3	508.0	0.06
Uranus	56.010	56.015	0.009	2563.6	2565.0	0.06
Neptune	94.020	94.030	0.011	4487.3	4489.0	0.04

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