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Hyperbolic Spline Solutions of Fractional Relaxation-Oscillation Equations and Fractional-Order Boundary Value Problems

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Abstract

In complex viscoelastic media, the two phenomenons stress relaxation and oscillation damping have frequently perceptible history and classical models can't accurately explained physical behaviors in which path is dependent. Latest research revealed that fractional derivative based models can distinguish such damping and complex relaxation. A hyperbolic spline-based numerical scheme is utilized in this study to obtain solutions of the fractional relaxation-oscillation equations (ROE). Additionally, solutions to fractional boundary value problems (BVPs) using hyperbolic splines are covered. Usually, smooth solution of a differential equation (DE) of fractional order cannot be expected and this constitute a question that how to get attainable order of convergence of numerical methods. The convergence behavior of the used numerical scheme is also examined in detail. To verify the robustness of the proposed approach, several numerical examples are provided that highlight its accuracy and computational efficiency.

Keywords: Relaxation Oscillation Equation, Fractional Boundary-condition Problem, Hyperbolic Spline-based scheme, Caputo sense Fractional Operators, Numerical error estimation.

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1. Introduction

Over recent decades, fractional DEs have gained substantial attention due to their ability to more precisely describe the dynamics of real-world physical systems. Fractional-order derivatives serve as powerful tools for capturing memory effects and hereditary characteristics inherent in many natural and engineered processes. As a result, fractional models are widely employed in diverse areas, including the analysis of electrical and mechanical properties of complex materials, the study of rock rheology, and numerous other scientific and engineering applications.

Fractional-order BVPs have gained substantial attention in recent years due to their ability to model physical processes that exhibit memory, hereditary characteristics, and anomalous diffusion behaviors that cannot be captured by classical integer-order models. Unlike traditional boundary value problems, Fractional BVPs incorporate fractional derivatives-most commonly in the Caputo or Riemann–Liouville sense,

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which enable a more accurate representation of long-range temporal interactions and spatial heterogeneity within complex systems. These problems arise naturally in diverse applications, including viscoelasticity, porous-media flow, biological population dynamics, anomalous transport, electrochemistry, and thermal diffusion in heterogeneous materials. The nonlocal nature of fractional operators makes the analysis and numerical treatment of such problems more challenging, yet essential for understanding system responses governed by power-law memory kernels. As a result, the development of efficient numerical schemes for solving Fractional BVPs has become an active area of research, offering valuable tools for scientists and engineers working with real-world models where conventional differential equations fail to provide sufficient accuracy.

Researchers are interested in fractional order control and systems due to the importance of fractional calculus in various fields. A classical fractional-order differential equation that captures the dynamic behavior of systems with memory effects can generally be expressed in the following form:

$$\Omega(\sigma, D^{\alpha_1}z, D^{\alpha_2}z, ..., D^{\alpha_n}z) = 0.$$

A relaxation oscillator is a system whose oscillatory behavior arises from the natural tendency of a physical process to restore itself to equilibrium after experiencing a disturbance [1,2]. Fractional derivatives, including positive fractional and fractal derivatives, frequently appear in the formulation of various relaxation oscillation models [3]. The fundamental equation governing oscillatory and relaxation-driven phenomena is referred to as the ROE. The fractional form of this relaxation model can be expressed as:

$$D^{\alpha}z(\sigma) + \widetilde{\phi_2}z(\sigma) = \omega(\sigma), \qquad \sigma > 0, \tag{1.1}$$

$$z(0) = 0, \qquad 0 < \alpha \leqslant 1,$$

or

$$D^{\alpha}z(\sigma) + \widetilde{\phi_2}z(\sigma) = \omega(\sigma), \qquad t > 0, \tag{1.2}$$

$$z(0) = \phi_1, \qquad z'(0) = \phi_2, \qquad 1 < \alpha \leqslant 2,$$

where $\widetilde{\phi_2}$ is a constant with positive value. This equation is referred to as the fractional-order ROE for $0 < \alpha \leqslant 2$. The model describes relaxation behavior characterized by power-law attenuation for $0 < \alpha < 1$. The model describes the oscillation of damping with damped viscoelastic intrinsic of oscillator for $1 < \alpha < 2$. Electrical heart model, signal processing, cardiac pacemakers modeling, system of predator-prey, interactions of spruce-budworm are some examples in which this model can be applied [3,4].

Anomalous diffusion processes and heat transfer in heterogeneous materials are examples of physical phenomena that can be modeled using fractional boundary value problems [5]. An intriguing and significant class of mathematical models consists of boundary value problems with integral boundary conditions. This paradigm can be applied to problems with multipoint, two-point, three-point, four-point, and other nonlocal boundary conditions. Cellular systems and population dynamics are two examples of applications where such integral-type boundary conditions naturally occur [6,7]. The model of a micro-electro-mechanical system (MEMS) equipment intended to assess the viscosity-fluids encountered during oil well exploration similarly exhibits these boundary value problems [8]. Understanding and describing many physical systems necessitates the solution of fractional-order DEs. It is not an easy task to get the analytic solutions of most fractional differential equations. Anyhow if we calculate the analytic solution, we can't utilize it because of its complex form. In this context, approximate solution is the appropriate way for the modeling of practical problems.It is evident that there is at least one solution to two-point boundary value issues of fractional order that can be viewed in [9-14].

In [15], fractional PDEs in Riesz space are solved by a hybrid B-spline collocation technique (HBCM), which transforms the fractional PDE into a system of linear algebraic equations. A Von Neumann stability study is part of it. In order to solve fractional Painlevé and Bagley-Torvik equations numerically

under various fractional-derivative definitions, this paper develops a hybrid collocation method that combines hyperbolic and cubic B-splines. It performs stability and convergence analysis, derives a banded algebraic system, and provides examples to demonstrate the accuracy and computing efficiency of the method [16]. In [17], time-fractional diffusion-wave equation (Caputo-Fabrizio derivative) is numerically solved using a uniform hyperbolic polynomial B-spline for spatial discretization along with a θ-weighted finite-difference scheme in time. Stability and convergence analyses demonstrate the effectiveness of this approach for fractional partial differential equations and boundary-value problems.
The structure of this article is presented as follows. Section 2 provides several fundamental results from fractional calculus. In Section 3, the consistency relations are derived using the hyperbolic spline function. Section 4 presents the numerical method for solving the Fractional BVPs , along with the matrix formulation of the scheme and its convergence analysis. Section 5 discusses the numerical procedure for the fractional ROE and establishes its convergence properties. Section 6 contains five numerical examples that demonstrate and compare the performance and efficiency of the proposed method.

2. Preliminaries

Let $z(\sigma)$ be a function defined on finite interval (σ,τ), then

Definition 2.1. [18,19]
In the $\mathit{Riemann\ Liouville}$ framework, the fractional derivative of order α is expressed as

$$^{R}D^{\alpha}z(\sigma)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{\rho}}{ds^{\rho}}\int_{0}^{\sigma}(\sigma-s)^{\rho-\alpha-1}z(s)ds,\ \alpha>0,\ \rho-1<\alpha<\rho,$$

The fractional integrals of order α, taken in the $\mathit{left\ and\ right\ sided\ Riemann\ Liouville}$ sense, are expressed as :

$$I^{\alpha}_{\sigma+}z(\sigma)={}^{R}D^{-\alpha}_{\sigma+}z(\sigma)=\frac{1}{\Gamma(\alpha)}\int_{\sigma}^{\sigma}(\sigma-s)^{\alpha-1}z(s)ds,\ \alpha>0$$

and

$$I^{\alpha}_{\tau-}z(\sigma)={}^{R}D^{-\alpha}_{\tau-}z(\sigma)=-\frac{1}{\Gamma(\alpha)}\int_{\tau}^{\sigma}(\sigma-s)^{\alpha-1}z(s)ds,\ \alpha>0,$$

respectively **[18-20]**.

Definition 2.2. [18-20]
The fractional derivatives of order α, taken in the $\mathit{left\ and\ right\ sided\ caputo}$ sense, are expressed as:

$$D^{\alpha}_{-\tau}z(\sigma)=\begin{cases} I^{\rho-\alpha}_{-\tau}D^{\rho}z(\sigma), & \rho-1<\alpha<\rho,\rho\in\mathbb{N},\\ \frac{D^{\rho}z(\sigma)}{D\sigma^{\rho}}, & \alpha=\rho\end{cases}$$

and

$$D^{\alpha}_{\sigma+}z(\sigma)=\begin{cases} I^{\rho-\alpha}_{\sigma+}D^{\rho}z(\sigma), & \rho-1<\alpha<\rho,\rho\in\mathbb{N},\\ \frac{D^{\rho}z(\sigma)}{D\sigma^{\rho}}, & \alpha=\rho,\end{cases}$$

respectively, here D^{ρ} is ordinary differential operator.

Key Characteristics of Fractional Integral and Derivative Operators [20-22]

1. If $\alpha,\ \mu>0$ and $z(\sigma)$ is continuous function, then the following properties hold:
   (i) $I^{\alpha}I^{\mu}z(t)=I^{\mu}I^{\alpha}z(\sigma)=I^{\alpha+\mu}z(\sigma)$
   (ii) $I^{\alpha}t^{\rho}=\frac{\Gamma(\rho+1)}{\Gamma(\rho+1+\alpha)}\sigma^{\rho+\alpha}$
   (iii) $^{R}D^{\alpha}(D^{-\mu}z(\sigma))={}^{R}D^{\alpha-\mu}z(\sigma)$

2. If $1 > \alpha,\ \mu > 0$ and $z(\sigma)$ is continuous, then the following properties hold:
   (i) $I^{\alpha}_{\tau-}I^{\mu}_{\tau-}z(\sigma) = I^{\mu}_{\tau-}I^{\alpha}_{\tau-}z(\sigma) = I^{\alpha+\mu}_{\tau-}z(\sigma)$
   (ii) $D^{\alpha}_{\tau-}I^{\alpha}_{\tau-}z(\sigma) = z(\sigma)$
   (iii) $D^{-\alpha}_{\tau-}(\tau-\sigma)^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1+\alpha)}(\tau-\sigma)^{\rho+\alpha}, \quad \rho \in \mathbb{N}.$
3. Let $\alpha \in \mathbb{R}$, $\rho - 1 < \alpha < \rho$, $\rho \in \mathbb{N}$, $j \in \mathbb{R}$.In the Caputo sense, the fractional derivative of an exponential function takes the following form for ρ

$$D^{\alpha}e^{j\sigma} = \sum_{k=0}^{\infty}\frac{j^{k+\rho}\sigma^{k+\rho-\alpha}}{\Gamma(k+1+\rho-\alpha)}.$$

4. $D^{\alpha}C = 0, \qquad C \text{ is constant}.$
5. $D^{\alpha}(\lambda z(\sigma) + \mu q(\sigma)) = \lambda D^{\alpha}z(\sigma) + \mu D^{\alpha}q(\sigma).$
6. $D^{\alpha}z(\sigma) =^{R} D^{\alpha}[z(\sigma) - \sum_{k=0}^{\rho-1}\frac{1}{k!}(\sigma-\sigma)^{k}z^{k}(\sigma)].$

3. Hyperbolic Spline Method

Let $\sigma_i = \sigma + il$ $(i = 0,1,...,n,\ \ l = \frac{\tau-\sigma}{n},\ n > 0)$ denote the grid points of a uniform partition of the interval $[\sigma,\tau]$ into subintervals $[\sigma_{i-1},\sigma_i]$. Let $z(\sigma)$ represent the exact solution of Eq.(1) and let S_i be an approximation to $z_i = z(\sigma_i)$ obtained using the spline function Υ_i which interpolates the points (σ_i, S_i) and (σ_{i+1}, S_{i+1}). Assume that, within each subinterval, the hyperbolic spline segment is defined as follows:

$$\Upsilon_i(\sigma) = \sigma_i \cosh k_1(\sigma_{i+1}-t) + \tau_i \sinh k_2(\sigma_{i+1}-t) + \xi_i, \qquad i = 0,1,...,n-1,$$

here σ_i, τ_i, ξ_i and d_i are undetermined coefficients, which can be written in terms of S_i and M_i as:

$$\Upsilon_i(\sigma_i) = S_i, \qquad \Upsilon_i(\sigma_{i+1}) = S_{i+1}, \qquad \Upsilon'_i(\sigma_{i+1}) = D_{i+1},$$

and are calculated, as

$$\sigma_i = \frac{S_{i+1}}{1-\cosh\theta_1} - \frac{S_i}{1-\cosh\theta_1} - \frac{D_{i+1}\sinh\theta_2}{k_2(1-\cosh\theta_1)}, \qquad \tau_i = -\frac{D_{i+1}}{k_2},$$

$$\xi_i = -\frac{(\cosh\theta_1)S_{i+1}}{1-\cosh\theta_1} + \frac{S_i}{1-\cosh\theta_1} + \frac{D_{i+1}\sinh\theta_2}{k_2(1-\cosh\theta_1)},$$

where $\theta_i = k_i l$ for $i = 1,2$. Using derivative continuities up to first order, along with the specified constant values, this leads to the following recurrence relation:

$$S_{i+1} - S_i = l(\gamma_1 D_i + \gamma_2 D_{i+1}), \qquad i = 0,1,...n-1, \tag{3.1}$$

where

$$\gamma_1 = \frac{-(1-\cosh\theta_1)}{\theta_1\sinh\theta_1},$$

$$\gamma_2 = \frac{\cosh\theta_2(1-\cosh\theta_1)}{\theta_1\sinh\theta_1} + \frac{\sinh\theta_2}{\theta_2}.$$

4. Solution Methodology for Fractional Boundary Value Equations

In this case, examine the following Fractional BVP:

$$D^{\alpha}z(\sigma)+z(\sigma)=\omega(\sigma), \qquad \sigma\in[0,1], \quad \rho-1<\alpha<\rho, \tag{4.1}$$

subject to

$$z(0)=\phi_1, \qquad z(1)=\phi_2. \tag{4.2}$$

here, ϕ_1 and ϕ_2 are real constants, $\omega(\sigma)$ is continuous function on the interval $[0,1]$ and D^{α} denotes the derivative in Caputo sense. In this case, D_i can be determine from the following equation,

$$D^{\alpha}\Upsilon_i(\sigma)\mid_{\sigma=\sigma_i}+S_i=\omega_i, \qquad\qquad i=0,1,...n, \tag{4.3}$$

where $\omega_i=\omega(\sigma_i)$.

Lemma 4.1. *Let $z\in C^3[\sigma,\tau]$ then the local truncation error σ_i , $i=0,1,...,n-1$, associated with the Eq. (3) is:*

$$\widetilde{\sigma}_i=\frac{-1}{12}l^3z^{(3)}(\sigma_i)+O(l^4), \tag{4.4}$$

Proof : To obtain the local truncation errors $\widetilde{\sigma}_i$, $i=0,1,...,n-1$ of Eq. (3), firstly rewrite this equation in the following form:

$$\widetilde{\sigma}_i=z_{i+1}-z_i-l[\gamma_1z_i^{(1)}+\gamma_2z_{i+1}^{(1)}], \qquad i=0,1,...,n-1,$$

The functions $z_i, z_i^{(1)}, z_i^{(2)}, z_i^{(3)}, z_i^{(4)}, z_i^{(5)}$ and $z_i^{(6)}$ can be expanded about the point t_i using Taylor's series. Following the same procedure, $\widetilde{\sigma}_i$, $i=0,1,...,n-1$, are obtained, as

$$\widetilde{\sigma}_i=lz_i^{(1)}(1-\gamma_1-\gamma_2)+l^2z_i^{(2)}(0.5-\gamma_2)+l^3z_i^{(3)}(\frac{1}{6}-\frac{\gamma_2}{2})+...$$

For $\gamma_1=\frac{1}{2}$, $\gamma_2=\frac{1}{2}$, the local truncation errors $\widetilde{\sigma}_i$ for $i=0,1,...,n-1$, are:

$$\widetilde{\sigma}_i=\frac{-1}{12}l^3z^{(3)}(\sigma_i)+O(l^4).$$

Moreover,

$$||T||_{\infty}=c_1l^3\Omega_3, \qquad \Omega_3=max_{\sigma\in[0,1]}|z^{(3)}(\sigma)|,$$

where c_1 is a constant and does not depend on l.

4.1. Matrix Formulation of the Proposed Scheme

Consider the $(n-1)$-dimensional column vectors: $Z=[z_1,z_2,...,z_{n-1}]^T$, $S=[S_1,S_2,...,S_{n-1}]^T$, $D=[D_1,D_2,...,D_{n-1}]^T$, $G=[G_1,G_2,...,G_{n-1}]^T$, $E=(e_i)$ and $T=(\widetilde{\sigma}_i)$ where $i=1,2,...,n-1$.
After substituting the values of D_i in the system (3), the system becomes in matrix notation as:

$$\phi_1S=G+G^* \tag{4.5}$$

where the vectors G, G^* and matrix ϕ_1 are presented in Appendix.

$$\phi_1Z=G+G^*+T. \tag{4.6}$$

From Eq.(8) and Eq.(9), the error equation can be obtained as,

$$\phi_1E=T. \tag{4.7}$$

Rewrite Eq.(10) as,

$$E=\phi_1^{-1}T. \tag{4.8}$$

4.2. Convergence of the Method

Lemma 4.2. *[21]*
If $\|\Omega\| < 1$, then the following exists:
1. $(I + \Omega)^{-1}$
2. $\|(I + \Omega)^{-1}\| < \frac{1}{1-\|\Omega\|}$*, where* Ω *is a matrix of order* n*.*

Lemma 4.3. *For* $m = 2$*,*

$$\|\phi_1^{-1}\|_\infty \leqslant \frac{-2(\mu_2(1-\cosh\theta_1)+\mu_1\sinh\theta_2)}{k_2(8(1-\cosh\theta_1)-2\mu_1(1-\cosh\theta_1)-2\mu_1\cosh\theta_1-2l(1-\cosh\theta_1)+l\mu_1)}, \tag{4.9}$$

satisfies the inequality provided that $\frac{2\mu_1\cosh\theta_1-l\mu_1}{(1-\cosh\theta_1)(8-2\mu_1-2l)} < 1$*, where* $\mu_i = k_i^m l^{m-\alpha} E_{1,m-\alpha+1}(k_i l) + (-k_i)^m l^{m-\alpha} E_{1,m-\alpha}$
for $i = 1,\ 2$*.*

Proof : The matrix ϕ_1 can be written, as

$$\phi_1 = I + \widetilde{\phi_1},$$

where

$$\widetilde{\phi_1} = \begin{pmatrix} \widetilde{\sigma_2} & \widetilde{\sigma_3} & & & & \\ \widetilde{\sigma_1} & \widetilde{\sigma_2} & \widetilde{\sigma_3} & & & \\ & & \ddots & & & \\ & & & \widetilde{\sigma_1} & \widetilde{\sigma_2} & \widetilde{\sigma_3} \\ & & & & \widetilde{\sigma_1} & \widetilde{\sigma_2} \end{pmatrix}.$$

while,

$$\begin{aligned} \widetilde{\sigma_1} &= l\gamma_1\frac{(1-\mu_{12})}{\mu_{22}}, \\ \widetilde{\sigma_2} &= -2 + l\gamma_1\frac{\mu_{11}}{\mu_{22}} + l\gamma_2\frac{(1-\mu_{12})}{\mu_{22}}, \\ \widetilde{\sigma_3} &= 1 + l\gamma_2\frac{\mu_{11}}{\mu_{22}}, \\ \mu_{11} &= \frac{\mu_1}{2}\left(1+\frac{\cosh\theta_1}{1-\cosh\theta_1}\right), \\ \mu_{12} &= \frac{\mu_1}{2(1-\cosh\theta_1)} \end{aligned}$$

and

$$\mu_{22} = \frac{-\mu_2}{2k_2} - \frac{\mu_1\sinh\theta_2}{2k_2(1-\cosh\theta_1)}.$$

The matrix ϕ_1^{-1} can be written, as

$$\phi_1^{-1} = (I + \widetilde{\phi_1})^{-1},$$

Using the Lemma 4.2, if

$$\|\widetilde{\phi_1}\|_\infty < 1, \tag{4.10}$$

then

$$\|\phi_1^{-1}\|_\infty \leqslant \frac{1}{1-\|\widetilde{\phi_1}\|_\infty},$$

where

$$\|\widetilde{\phi_1}\|_\infty = -1 + l\frac{\mu_{11}}{\mu_{22}} + \frac{l(1-\mu_{12})}{2\mu_{22}}. \tag{4.11}$$

In simplify form,

$$\|\phi_1^{-1}\|_\infty \leqslant \frac{-2(\mu_2(1-\cosh\theta_1)+\mu_1\sinh\theta_2)}{k_2(8(1-\cosh\theta_1)-2\mu_1(1-\cosh\theta_1)-2\mu_1\cosh\theta_1-2l(1-\cosh\theta_1)+l\mu_1)}.$$

Lemma 4.4. *The matrix ϕ_1 in Eq. (10) is nonsingular, provided that:*

$$\frac{2\mu_1\cosh\theta_1 - l\mu_1}{(1-\cosh\theta_1)(8-2\mu_1-2l)} < 1,$$

Then

$$\|E\|_\infty \leqslant \|\phi_1^{-1}\|_\infty \|T\|_\infty \cong O(l^{4-\alpha}). \tag{4.12}$$

Theorem 4.5. *Let $z(t)$ be the analytical solution of the fractional order BVP Eq. (4) having boundary condition Eq. (5) and z_i satisfy the discrete BVP Eq. (9).*
Also, if $e_i = z_i - S_i$, then we have

$$\|E\|_\infty = O(l^{4-\alpha}),$$

where $i = 0, 1, 2, ..., n-1$.

5. Numerical Method for Fractional ROE

In second case, consider the following fractional ROE:

$$D^\alpha z(\sigma) + z(\sigma) = \omega(\sigma), \qquad \sigma > 0, \;\; 1 < \alpha < 2, \tag{5.1}$$

subject to

$$z(0) = \phi_1^*, \qquad z'(0) = \phi_2^*. \tag{5.2}$$

where ϕ_1^* and ϕ_2^* are real constants. Also, D^α denotes fractional derivative in Caputo's sense. For $i = 1, 2, ..., n$, the matrices T and Q are extracted from system (1). Consider n dimensional column vectors: $Z = [z_1, z_2, ..., z_n]^T$, $S = [S_1, S_2, ..., S_n]^T$, $D = [D_1, D_2, ..., D_n]^T$, $E = (e_i)$ and $T = (\widetilde{\sigma}_i)$ for $i = 1, 2, ..., n$.
From system (3), we have

$$TS = lQD + C, \tag{5.3}$$

where T, Q are $n \times n$ matrices and

$$Q = \frac{1}{6}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & \\ & & & & \ddots & & \\ & & & & & \frac{1}{2} & \frac{1}{2} & \\ & & & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & & \ddots & & & \\ & & & -1 & 1 & & \\ & & & & -1 & 1 & \\ & & & & & -1 & 1 \end{pmatrix}$$

and C is n dimensional column vector and

$$C = [l\gamma_1 D_0 + S_0, 0, ..., 0].$$

The system (6) in matrix form can be written, as

$$PS + HD = \Omega, \tag{5.4}$$

where P, H are matrices of order $n \times n$ and

$$P = \begin{pmatrix} \mu_{11} & & & & \\ 1-\mu_{12} & \mu_{11} & & & \\ & & \ddots & & \\ & & & & \\ & & & 1-\mu_{12} & \mu_{11} \end{pmatrix},$$

$$H = \begin{pmatrix} \mu_{22} & & & & \\ & \mu_{22} & & & \\ & & \ddots & & \\ & & & & \\ & & & & \mu_{22} \end{pmatrix}.$$

Moreover $\Omega = (\omega_i)$ is a column vector with n components subject to

$$F = \begin{cases} \omega_0 - (1-\mu_{12})S_0, & i = 1, \\ \omega_i, & i = 2, 3, ..., n. \end{cases}$$

The Eq. (19) is equivalent to

$$D = H^{-1}\Omega - H^{-1}PS.$$

From Eq. (18) and Eq. (19), we have

$$(T + lQH^{-1}P)S = lQH^{-1}\Omega + C \tag{5.5}$$

5.1. Convergence Analysis of the Method

For error bound on, consider

$$(T + lQH^{-1}P)Z = lQH^{-1}\Omega + C + T. \tag{5.6}$$

BY the Eq. (20) and Eq. (21),

$$(T + lQH^{-1}P)E = T. \tag{5.7}$$

Using Eq. (22), E can be shown as

$$E = (I + lT^{-1}QH^{-1}P)^{-1}T^{-1}T. \tag{5.8}$$

Lemma 5.1.

$$\|H^{-1}\|_\infty \leqslant \frac{2k_2(1-\cosh\theta_1)}{(4k_2+\mu_2)(1-\cosh\theta_1)+\mu_1\sinh\theta_2}, \tag{5.9}$$

satisfies the inequality provided that $\frac{(4k_2+\mu_2)(1-\cosh\theta_1)}{\mu_1\sinh\theta_2} < 1$.

Proof : The matrix H takes the form

$$H = I + \widetilde{H},$$

where $\widetilde{H}$:

$$\begin{pmatrix} h_1^* & & & & & \\ & h_1^* & & & & \\ & & h_1^* & & & \\ & & & \ddots & & \\ & & & & h_1^* & \\ & & & & & h_1^* \end{pmatrix}.$$

Also,

$$h_1^* = \frac{-\mu_2}{2k_2} - \frac{\mu_1 \sinh\theta_2}{2k_2(1-\cosh\theta_1)} - 1.$$

Consider,

$$H^{-1} = (I + \widetilde{H})^{-1}.$$

By Lemma 4.2, if

$$\|\widetilde{H}\|_\infty < 1,$$

then

$$\|H^{-1}\|_\infty \leqslant \frac{1}{1 - \|\widetilde{H}\|_\infty}, \tag{5.10}$$

where

$$\|\widetilde{H}\|_\infty = h_1^*.$$

After simplification, Eq. (25) can be written as,

$$\|H^{-1}\|_\infty \leqslant \frac{2k_2(1-\cosh\theta_1)}{(4k_2+\mu_2)(1-\cosh\theta_1) + \mu_1 \sinh\theta_2}.$$

Lemma 5.2. *The matrix* $(T + lQH^{-1}P)$ *in Eq. (20) is nonsingular, provided that:*

$$\frac{\lambda_2}{2\lambda_1\mu_{11}k_2(1-\cosh\theta_1)} < 1,$$

where $\lambda_1 = b - a$, $\lambda_2 = (4k_2+\mu_2)(1-\cosh\theta_1) + \mu_1\sinh\theta_2$. *Then*

$$\|E\|_\infty \leqslant \frac{c_1 Z_3 l^2 \lambda_1 \lambda_2}{\lambda_2 - 2\lambda_1\mu_{11}k_2(1-\cosh\theta_1)} \cong O(l^2). \tag{5.11}$$

Proof : From Lemma 4.2,

$$\|E\|_\infty = max_{1\leqslant i\leqslant n}|e_i| \leqslant \frac{\|T^{-1}\|_\infty \|T\|_\infty}{1 - l\|T^{-1}\|_\infty \|Q\|_\infty \|H^{-1}\|_\infty \|P\|_\infty}, \tag{5.12}$$

provided that $l\|T^{-1}\|_\infty \|Q\|_\infty \|H^{-1}\|_\infty \|P\|_\infty < 1$. As,

$$\|T^{-1}\|_\infty = \frac{\tau - \sigma}{h}.$$

Table 1: Max Absolute Errorsfor $\alpha = 1.4$.

h	k_1	k_2	$Maximum\ absolute\ error$
$1/8$	$1.0E-03$	-1	$6.64E-01$
$1/16$	-23	0.9	$9.4E-02$
$1/32$	-100	-1	$5.3E-02$
$1/64$	150	-10	$6.5E-03$

Table 2: Max Absolute Errorsfor $\alpha = 1.7$.

h	k_1	k_2	$Maximum\ absolute\ error$
$1/8$	$1.0E-03$	-1	$5.3E-01$
$1/16$	1	-1	$4.57E-02$
$1/32$	-40	-0.9	$5.31E-02$
$1/64$	130	1	$9.9E-03$

Also,

$$\|Q\|_\infty = 1$$

and

$$\|P\|_\infty = \mu_{11}.$$

Using $\|T^{-1}\|_\infty, \|P\|_\infty, \|H^{-1}\|_\infty$ and $\|Q\|_\infty$ in Eq. (27),

$$\|E\|_\infty \leqslant \frac{c_1 Z_3 l^2 \lambda_1 \lambda_2}{\lambda_2 - 2\lambda_1 \mu_{11} k_2 (1 - \cosh\theta_1)} \cong O(l^2). \tag{5.13}$$

Theorem 5.3. *Let $z(\sigma)$ be analytical solution of the fractional order DE Eq. (16) having initial condition Eq. (17) and z_i, be the solution of the discrete IVP Eq. (21). Also, if $e_i = z_i - S_i$, then*

$$\|E\|_\infty = O(l^2),$$

where $i = 0, 1, 2, ..., n-1$.

6. Numerical Illustrations

Example 6.1 Examine the following Fractional BVPs:

$$D^\alpha z(\sigma) + z(\sigma) = \omega(\sigma), \qquad \sigma \in [0, 1],$$

with

$$z(0) = 0, \ \ z(1) = 0.5,$$

where $\omega(\sigma) = \sigma^4 - \frac{\sigma^3}{2} - \frac{3\sigma^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{24\sigma^{4-\alpha}}{\Gamma(5-\alpha)}$. The analytical solution of the problem is $\sigma^4 - \frac{\sigma^3}{2}$. This spline based technique is applied for various values of α and extracted findings are shown in Figure 1 and Table 1, 2, 3.

Example 6.2 Examine the following fractional ROE:

$$D^\alpha z(\sigma) + z(\sigma) = f(\sigma), \qquad 1 < \alpha < 2,$$

with

$$z(0) = 1, \ \ z'(0) = 0,$$

Table 3: Max Absolute Errorsfor $\alpha = 1.9$.

h	k_1	k_2	Maximum absolute error
1/8	$1.0E-03$	-1	$1.4E-01$
1/16	$1.0E-03$	-1	$8.56E-02$
1/32	$1.0E-03$	-1	$3.79E-02$
1/64	$1.0E-03$	-1	$2.5E-03$

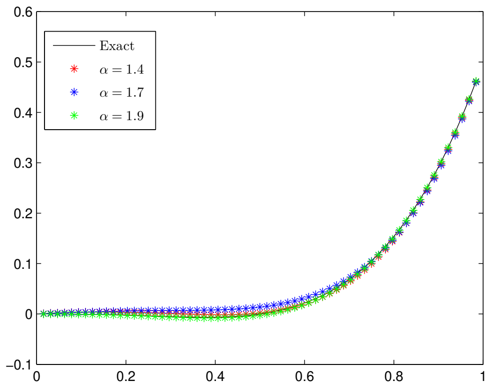

Figure 1: Analytical and Numerical Solutions of Example 1 with different value of α.

where $\omega(t) = 0$. The analytical solution of the problem is $E_{\alpha,1}(-\sigma^{\alpha})$. This scheme is applied for $\alpha = 1.3$, $k_1 = 0.0001$ and $k_2 = -3.39$. The results are provided in Table 4.

Example 6.3 Examine the following IVP:

$$D^{\alpha}z(\sigma) + z(\sigma) = f(\sigma), \qquad 1 < \alpha < 2,$$

with

$$z(0) = 1, \;\; z'(0) = 0,$$

Table 4: Numerical results of presented method for $\alpha = 1.3$.

t	Exact Solution	Approx. Solution	Maximum absolute Errors
0.1	0.9577	0.9836	2.59E − 002
0.2	0.8982	0.9391	4.09E − 002
0.3	0.8321	0.8755	4.33E − 002
0.4	0.7631	0.7998	3.67E − 002
0.5	0.6932	0.7176	2.44E − 002
0.6	0.6239	0.6334	9.5E − 003
0.7	0.5562	0.5505	5.7E − 003
0.8	0.4908	0.4712	1.96E − 002
0.9	0.4282	0.3972	3.11E − 002
1	0.3689	0.3296	3.94E − 002

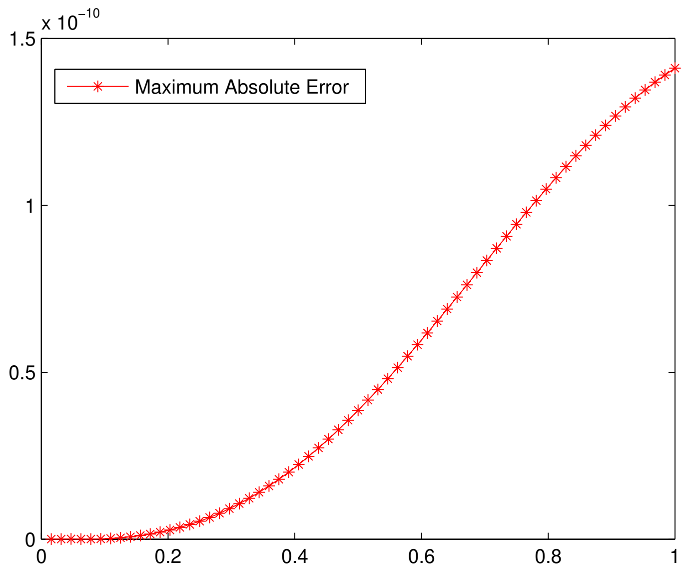

Figure 2: Absolute Error with $\alpha = 1.5$.

where $\omega(\sigma) = \cos(\lambda\sigma) - 0.5\lambda^2\sigma^{1.33}(E_{1,2.333}(i\lambda t) + E_{1,2.333}(-i\lambda\sigma))$. The analytical solution of the problem is $\cos(\lambda\sigma)$. This spline based technique is applied for $\alpha = 1.5$. The findings are presented in Table 5 and Figure 2.

Example 6.4 Examine the following initial value problem:

$$D^{\alpha}z(\sigma) + z(\sigma) = \omega(\sigma),$$

with

$$z(0) = 0, \quad z'(0) = 0,$$

where $\omega(\sigma) = \sigma^2 - \frac{2\sigma^{2-\alpha}}{\Gamma(3-\alpha)}$. The analytical solution of the problem is σ^2. This scheme is applied for different values of α and results are shown in Table 6, 7, 8 and Figure 3.

Example 6.5 Examine the following boundary value problem:

$$D^{\alpha}z(\sigma) + z(\sigma) = \omega(\sigma), \qquad \sigma \in [0,1], \; 1 < \alpha < 2,$$

with

$$z(0) = 0, \quad z(1) = 0,$$

Table 5: Max Absolute Errors for $\alpha = 1.5$.

h	k_1	k_2	e	$Maximum\ absolute\ error$
1/8	$1.0E-02$	$1.0E-02$	$1.0E-02$	$1.4040E-005$
1/16	$1.0E-03$	$1.0E-03$	$1.0E-03$	$4.2977E-007$
1/32	$1.0E-04$	$1.0E-04$	$1.0E-04$	$7.4715E-009$
1/64	$1.0E-05$	$1.0E-05$	$1.0E-05$	$1.0749E-010$

Table 6: Max Absolute Errorsfor $\alpha = 0.5$.

h	k_1	k_2	$Maximum\ absolute\ error$
1/8	$-1.0E-03$	-7.99	$1.3204E-01$
1/16	270	-17	$9.54E-02$
1/32	325	-50	$5.59E-02$

Table 7: Max Absolute Errorsfor $\alpha = 0.9$.

h	k_1	k_2	$Maximum\ absolute\ error$
1/8	$1.0E-06$	-5	$1.331E-01$
1/16	40	-10	$8.62E-02$
1/32	271	-18	$5.79E-02$

Table 8: Max Absolute Errorsfor $\alpha = 1.5$.

h	k_1	k_2	$Maximum\ absolute\ error$
1/8	$1.0E-03$	-1	$1.106E-01$
1/16	15	-2	$6.71E-02$
1/32	101	-0.5	$7.53E-02$

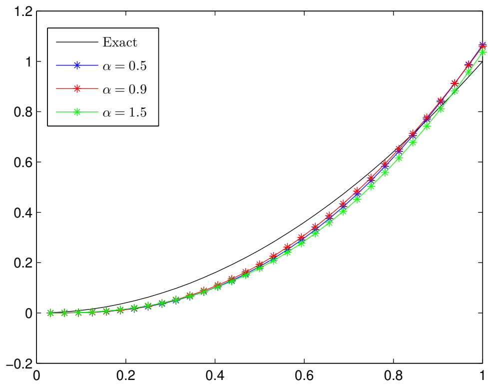

Figure 3: Analytical and Numerical Solutions of Example 2 with different value of α.

The analytical solution of the problem is $\sigma^5 - \sigma^4$. This scheme is applied for $\alpha = 1.99$, $k_1 = 0.0001$ and $k_2 = -1$. The results are shown in Table 9.

Table 9: Max Absolute Errorsfor $\alpha = 1.99$.

h	$Maximum\ absolute\ error$
$1/8$	$4.44E-002$
$1/16$	$2.24E-002$
$1/32$	$9.9E-003$

Appendix
The matrix ϕ_1 and the vectors G, G^* are given in Eq. (8) are:

$$\phi_1 = \begin{pmatrix} \sigma_2 & \sigma_3 & & & & \\ \sigma_1 & \sigma_2 & \sigma_3 & & & \\ & & \ddots & & & \\ & & & \sigma_1 & \sigma_2 & \sigma_3 \\ & & & & \sigma_1 & \sigma_2 \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{l}{\mu_{22}}(\gamma_1 f_0 + \gamma_2 f_1) \\ \frac{l}{\mu_{22}}(\gamma_1 f_1 + \gamma_2 f_2) \\ \vdots \\ \frac{l}{\mu_{22}}(\gamma_1 f_{n-3} + \gamma_2 f_{n-2}) \\ \frac{l}{\mu_{22}}(\gamma_1 f_{n-2} + \gamma_2 f_{n-1}) \end{pmatrix}$$

and

$$G^* = \begin{pmatrix} -\frac{l}{\mu_{22}}\gamma_1(1-\mu_{12}) * S_0 \\ 0 \\ \vdots \\ 0 \\ -(1+\frac{l}{\mu_{22}}\mu_{11}\gamma_1) * S_n \end{pmatrix}.$$

Where $\omega_i = \omega(\sigma_i)$, $\sigma_2 = \widetilde{\sigma_2} + 1$, $\sigma_3 = \widetilde{\sigma_3}$ and $\sigma_1 = \widetilde{\sigma_1}$.

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