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Stability of Einstein Static Universe within f(R, ϕ) Framework

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Abstract

The aim of this work is to discuss the stability issues of the Einstein static cosmos in $f(R, \phi)$ theory, where R correspond to Ricci scalar and ϕ a scalar field. To this end, we considered a spherically symmetric spacetime and apply homogeneous scalar perturbations to the unperturbed field equations within this modified framework. The regions having stable solutions are determined by equation of state $p = \omega\rho$ and we linearized the perturbed field equations for two distinct models of the $f(R, \phi)$ theory. In contrast to special theory of general relativity, we observe that, for $f(R, \phi)$ theory, Einstein universe is stable with (stability) cosmological constant exists. Consequently, the solutions which are not stable in general relativity can behave as stable due to the extensions of $f(R, \phi)$ theory.

Keywords: Modified Theories, Exact Solutions, Perturbation Theory.

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1. Introduction

Recent advancements in cosmology and astronomy have drastically altered our understanding of the universe. Observations from Type Ia Supernovae, along with measurements of the Cosmic Microwave Background (CMB) and results from the BICEP experiment, have all pointed toward a remarkable conclusion: the universe is not just expanding, but doing so at an accelerating pace [1]–[3]. This surprising acceleration of the Universe is usually explained by introducing an unknown entity called dark energy. Despite its central role in cosmic evolution, the fundamental origin and physical nature of dark energy remain among the most significant open problems in contemporary cosmology.

From observational stand point; the Planck mission (2018) [4], Sloan Digital Sky Survey [5] and Wilkinson Microwave Anisotropy Probe [6] evidence, has greatly helped to improve our understanding with respect to energy content of the Universe. Based upon these measurements, it follows that 68.3 percent of the cosmic energy density is comprised by dark energy, 26.8 percent by dark matter, and only about

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4.9 percent consists of ordinary baryonic matter-stuff which interacts with electromagnetic or strong force responsible for stars, planets and galaxies formation.

Driven by these observational results, the latest cosmological studies have also paid more and more attention on modified gravity theories as feasible alternative explanations for cosmic acceleration and realization of darkenergy. These are theories which deviate from the usual (or Einstein–Hilbert) description of gravity by including extra functions or fields to generalize the gravitational sector to partially describe phenomena that cannot be described entirely underGeneral Relativity. Among these models the $f(R)$ gravity model has garnered much interest in theliterature, in which the Ricci scalar R is replaced by a general function of R. Such a modification yields a richer theory of gravity and further, offers the possibility of one unified theory that would be capable to account both for the very early Universe inflation and also for the present late time accelerated expansion [7]-[8].

In order to go beyond the holography, we would need to generalize our description of dark energy. $f(R, \phi)$ theory is an extension of gravitational dynamics involving a scalar field and the Ricci scalar(R). The additional degree of freedom increases the model variety, especially in terms of providing a theory for the late time acceleration of the cosmos [9]. The action of ϕ plays a crucial role to control the cosmic expansion history and provides an attractive mechanism for explaining recent observation of accelerated expansion.

In addition, the $f(R, \phi)$ model introduces potentially nonminimal coupling between the scalar field and matter, which can cause departures from general-relativistic behavior on cosmic scales. The possible nature of the interactions give useful information about its physical properties, like as on the dark energy and also regarding their influence on Large Scale Structure (LSS) formations, such as galaxy or cluster individualies [10]. By extending the gravitational action in this way, it covers a wider range of accelerated Universe than conventional gravity as well as standard modified gravity models.

Malik [11] examined cylindrical symmetric solutions within the $f(R, \phi)$ gravity framework, using the specific model $f(R, \phi) = (1 + \xi\phi^2)R$. In this work, the energy conditions particularly the null energy condition were analyzed, and its violation in certain regions was interpreted as evidence for the possible existence of cylindrical wormholes. In a related study, Malik [11] also explored the structure of charged compact stars in the $f(R, \phi)$ scenario, employing the Krori–Barua metric to obtain exact solutions of the field equations. Asghar *et al.* [13] applied the Karmarkar condition within the $f(R, \phi)$ framework to analyze compact stars, confirming the models viability and the physical accuracy of the observed anisotropic findings. Farajollahi *et al.* [14] studied FRW cosmology in $f(R, \phi)$ theory, showing that in certain cases the equation of state parameter crosses the phantom divider, and identifying model parameters that satisfy the independent tests of Cosmological Redshift Drift and type Ia supernova luminosity distances. Thus, it appears intriguing to further explore the $f(R, \phi)$ theory.

The Einstein Static Universe is a cosmological model in which the universe remains static, neither expanding nor contracting, due to a precise balance between the gravitational attraction of matter and the repulsive force from a positive cosmological constant. However, this solution to Einstein's field equations is inherently unstable in the context of classical general relativity (GR), as any small perturbation could cause the universe to either collapse or expand. Despite this, static Einstein Universe stability has been a key focus in modified gravity theories, where it has been suggested that certain modifications might stabilize this solution.

The main focus of this work is to investigate the stability issues of the Static Einstein universe in $f(R, \phi)$ framework. In Section **2**, we discuss the field equations of $f(R, \phi)$ gravity and apply linear perturbation methods to both the field equations and the spacetime. We then analyze the stability of the static Einstein cosmos under perturbations in both strong and weak gravitational fields. The main outcomes of this work, including the stability regions and implications for the Einstein static universe, are discussed in Section **3**.

2. $f(R,\phi)$ Gravity and Einstein Static Cosmos

The general action for $f(R,\phi)$ gravity is given as [10]

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}[f(R,\phi) + w(\phi)\phi_{;\mu}\phi^{;\mu}] + L_m, \tag{1}$$

where $f(R,\phi)$ is a function of Ricci Scalar R and the scalar field ϕ, w is the scalar field function. From this action, we get the following Einstein equations,

$$FR_{\mu\nu} - \frac{1}{2}[f + w(\phi)\phi_{;\alpha}\phi^{;\alpha}]g_{\mu\nu} + w(\phi)\phi_{;\mu}\phi_{;\nu} - F_{;\mu\nu} + g_{\mu\nu}\Box F = T_{\mu\nu}, \tag{2}$$

where $F = \frac{d}{dR}f(R,\phi)$. We can retrieve $f(R)$ field equations by replacing $f(R,\phi)$ with $f(R)$. The trace of Eq.(2) is given as

$$FR - 2f - w(\phi)\phi_{;\alpha}\phi^{;\alpha} + 3\Box F = T. \tag{3}$$

We consider the FRW universe model, which is given as

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{4}$$

For Einstein static universe, we take $a(t) = a_0 = constant$ in spacetime as well as in field equation. The Ricci scalar becomes $R = 6/a_0^2$ and the Einstein equations takes the form:

$$\rho_0 = \frac{f}{2}, \; p_0 = \frac{2F}{(a_0)^2} - \frac{f}{2} - F'', \tag{5}$$

with ρ_0 and p_0 as the background energy density and pressure. Now, In this analysis, our method involves examining particular functional forms of $f(R,\phi)$ in order to systematically determine the stability characteristics of the Einstein static universe (ESU), specifically against linear homogeneous scalar perturbations. This stability assessment is performed around the background solution defined by eq. (5). To achieve this, we introduce disturbances or perturbations into both the energy density ($\delta\rho$) and the metric scale factor (δa). Crucially, these specific perturbations are assumed to depend solely on time, thereby simplifying the stability analysis by focusing exclusively on uniform (homogeneous) fluctuations.,

$$\rho(t) = \rho_0(1 + \delta\rho(t)), \; a(t) = a_0(1 + \delta a(t)), \; \phi(t) = \phi_0(1 + \delta\phi(t)). \tag{6}$$

We assume a linear EoS, $p(t) = \omega\rho(t)$, linearizing the perturbed field equations and dissecting the solutions.

3.1. $f(R,\phi) \propto R + R^2 + \phi$

Consider the following case,

$$f(R,\phi) = R + \frac{a_o^4}{6}(\sigma\alpha^2R^2) - 2\Lambda + \phi, \tag{7}$$

where a positive parameter α and $\sigma = \pm 1$. We presented the factor $\frac{a_0^4}{6}$ to impressively solve the equations, resulting in the examination laid below. The unperturbed field equation (5) for this model takes the following form

$$\rho_0 = \frac{3}{a_0^2} + 3\sigma\alpha^2 - \Lambda + \phi_0, \tag{8}$$

$$p_0 = \frac{-1}{a_0^2} + \sigma\alpha^2 + \Lambda - \phi_0, \tag{9}$$

Here Λ is defined as:

$$\Lambda = \frac{1}{2}\rho_0(1+3\omega) - 3\sigma\alpha^2 + \phi_0. \tag{10}$$

Then we obtain an equation for the evolution of the perturbation of scale factor (δa) in a self consistent way afterwards. The above disturbances we are studying in what follows-unleashed by Eq. (6), are formally induced in the metric quantities and energy tensor. In the approach, the perturbed equations of motion are Eq. (2). We then subtract the original, unperturbed background equations Eqs. (8) and (9). This subtraction step ensures that only terms linear in the small perturbations are left, namely the linearized equations of motion. Once this algebraic reduction is done we get the expression derived from, by using the (tt)-component of the field equations:

$$\delta\rho(t) = \frac{\phi_0}{2\rho_0}\delta\phi - 3(\omega+1)\delta a(t), \tag{11}$$

which are used to obtain the following component:

$$\begin{aligned}&[8\sigma\alpha^2 - \rho_0(1+3\omega)(1+\omega)][-4\sigma\alpha^2 + \rho_0(1+\omega)]^2\delta a(t) + 2[-\rho_0(1+\omega) + 8\alpha^2\sigma]\\ &\times[\rho_0(1+\omega) - 4\sigma\alpha^2]\delta a''(t) + 8\sigma\alpha^2\delta a^4(t) + \frac{\phi_0}{2}(1+\omega)[\rho_0(1+\omega) - 4\sigma\alpha^2]^2\\ &\times\delta\phi(t) = 0.\end{aligned} \tag{12}$$

In the limit, $\alpha \to 0$, above equation simplifies to

$$2\delta a''(t) - \delta a(t)\rho_0(1+3\omega)(1+\omega) + \frac{\phi_0}{2}\delta\phi(t)(1+\omega) = 0, \tag{13}$$

which gives the following results:

$$\delta a(t) = C_1 e^{\lambda t} + C_2 e^{-\lambda t},\ \delta\phi = C_3 e^{\xi t}, \tag{14}$$

where C_1, C_2 and C_3 are integration constant and λ as well as ξ are defined as

$$\lambda = \sqrt{\frac{1}{2}\rho_0(1+3\omega)(1+\omega)},\ \xi = (\omega+1). \tag{15}$$

To prevent the solution from either growing exponentially or collapsing, it remains stable within the range

$$\frac{-1}{3} > \omega > -1. \tag{16}$$

This particular range of parameters violates the Strong Energy Condition (SEC), a general requirement in General Relativity that says $\rho + 3p \geqslant 0$ for any energy density ρ and pressure p. This violation is important in that, just as in the classical Einstein universe, there will be an effective cosmological constant (Λ) that the static solution critically depends on :

$$\Lambda = \frac{1}{2}\rho_0(1+3\omega) + \phi_0, \tag{17}$$

and considering only positive energy densities, it follows that Λ becomes less then zero in the stability region.

The full solution to the modified perturbation differential equation, as given by Eq.(12), is expressed as

$$\delta a(t) = C_1 e^{\lambda_1 t} + C_2 e^{-\lambda_1 t} + C_3 e^{\lambda_2 t} + C_4 e^{-\lambda_2 t}, \delta\phi = C_5 e^{\xi t}, \tag{18}$$

where C_i (with $i = 1...5$) are constants. The expressions of λ_1, λ_2 and ξ are given by

$$\lambda_{1,2} = \Bigg[\frac{\rho_0 - 4\sigma\alpha^2}{8\sigma\alpha^2}[8\sigma\alpha^2 - \rho_0(1+\omega) \pm\sqrt{\rho_0(\omega+1)(\rho_0(\omega+1)+8\sigma\alpha^2(-1+3\omega))}]\Bigg]^{\frac{1}{2}}, \tag{19}$$

$$\xi = (1+\omega), \tag{20}$$

respectively.

For the subsequent analysis, a key constraint we impose is that the cosmological constant (Λ) must be positive ($\Lambda > 0$). When considering the case where the curvature parameter is $\sigma = -1$ (representing a negatively curved spatial geometry) alongside a positive cosmological constant, our investigation reveals that no stable solutions for the Einstein static universe can be established. However, a vastly different outcome emerges for the case where $\sigma = +1$ (a positively curved spatial geometry) and $\Lambda > 0$; under these conditions, the model admits three distinct regions of stability.

First Stability Region

$$\textbf{2.1A} \quad 8\alpha^2 < \rho_0 < \frac{3(7+\sqrt{17})\alpha^2}{2}, \tag{21}$$

$$\frac{8\alpha^2 - \rho_0}{24\alpha^2 + \rho_0} \leqslant \omega < \frac{1}{3}\left(-2 + \sqrt{\frac{24\alpha^2 + \rho_0}{\rho_0}}\right). \tag{22}$$

Second Stability Region

$$\textbf{2.1B} \quad \rho_0 = \frac{3(7+\sqrt{17})\alpha^2}{2}, \tag{23}$$

$$-\frac{5+3\sqrt{17}}{3(23+\sqrt{17})} < \omega < \frac{1}{3}\left(-2 + \sqrt{\frac{23+\sqrt{17}}{7+\sqrt{17}}}\right). \tag{24}$$

The inequalities simplify to $-0.213 < \omega < -0.146$.

Third Stability Region

$$\textbf{2.1C} \quad \rho_0 > \frac{3(7+\sqrt{17})\alpha^2}{2}, \tag{25}$$

$$\frac{6\alpha^2 - \rho_0}{3\rho_0} \leqslant \omega < \frac{1}{3}\left(-2 + \sqrt{\frac{24\alpha^2 + \rho_0}{\rho_0}}\right). \tag{26}$$

The important fact that given results are stable with the stability condition $f_{RR} = d^2f/dR^2 > 0$ of cosmological models at high curvatures [19].

Figure 1: Stability Region for $f \propto R + R^2 + \phi$

In Fig. 1 one can find (as 1) a graphical representation of the stability region for that particular functional coming from $f(R,\phi) = R + R^2 + \phi$. The graph is especially revealing: the solid line is a reference, which shows the region of pure General Relativity (GR) stability, and it vividly displays how the boundary of these regions changes when we take into account also the scalar field ϕ. For the lower patch of the graph, which as g->0 appropxiomately recovers GR, the effective cosmological constant (Λ) is counterintuitively found to be negative. On the other hand, upper triangular-like shaped region (denoted as 2.1A, 2.1B and 2.1C) shows the stability region of Einstein static universe (ESU) with positive Λ. This essential positive stability results from the fact that the scalar field (ϕ) as a quintessence modifies effectively to the equation of state parameter ω so, in this way it supports that universe everywhere by providing some effective negative pressure. This implications taken together imply that the scalar field bring about a much more flexible and stable structural framework for ESU compared to those already embedded in GR.

3.2. $f(R,\phi) \propto R + \frac{1}{R} + \phi$

Now, we consider

$$f(R,\phi) = R + \frac{\sigma\mu^4}{a_0^2}\frac{1}{R} - 2\Lambda + \phi, \tag{27}$$

where $\mu \geqslant 0$ and $\sigma = \pm 1$. As in the case I, we were looking for the factor a_0^2, which helps us to get the calculations in simplified form.

$f(R,\phi)$ considered in this work has been inspired one (like several others modified gravity models) by its ability to realize the present late-time accelerated expansion of the universe and it involves the scalar field (ϕ). The presence of scalar field ϕ at the end of the functional form does not change form so much at highive curvature limit. But it provides an extra contribution, which has relevance in the cosmological dynamics of the Universe. Unlike the "pure" $f(R)$ gravity, in which low-curvature behaviour is ruled by the $\frac{1}{R}$ term, here it turns out to be a scalar field ϕ a central ingredient for defining gravitational interaction at large scales.

Although the model has shown potential for describing a stable solution, it is important to note that in previous works, similar models in $f(R)$ gravity have suffered from instabilities under certain conditions [20]. However, in the case of $f(R,\phi)$ gravity, the additional scalar field can modify the stability conditions, leading to a more stable solution under appropriate conditions. This is evident in the fact that when modified, the term $f_{RR} = \frac{d^2f}{dR^2} > 0$ ensures the stability of the solution [19].

Assuming this scenario with the background equations (5) reduce to:

$$\rho_0 = \frac{3}{a_0^2} + \frac{\sigma\mu^4}{12} - \Lambda + \frac{1}{2}\phi_0, \tag{28}$$

$$p_0 = \frac{-1}{a_0^2} - \frac{5\sigma\mu^4}{36} + \Lambda - \frac{1}{2}\phi_0, \tag{29}$$

which gives that the cosmological constant as follows:

$$\Lambda = \frac{1}{2}\rho_0(1+3\omega) + \frac{\sigma\mu^4}{6} + \frac{1}{2}\phi_0. \tag{30}$$

Applying linear perturbation theory, as step forward on the similar patterns as previous case, we derive the following equation

$$\begin{aligned}&[2\sigma\mu^4 - 9\rho_0(1+\omega)(1+3\omega)][4\sigma\mu^4 + 18\rho_0(1+\omega)]^2\delta a(t) - 36[2\sigma\mu^4\\&-9\rho_0(1+\omega)][4\sigma\mu^4 + 18\rho_0(1+\omega)]\delta a''(t) + 648\sigma\mu^4\delta a^4(t) - \frac{\phi}{2}(1+\omega)\\&\times[4\sigma\mu^4 + 18\rho_0(1+\omega)]^2\delta\phi(t) = 0.\end{aligned} \tag{31}$$

The limit μ approaches to 0 this equation becomes the general relativistic one. Now, Equation (31) simplifies to

$$2\delta a''(t) - \rho_0(1+\omega)(1+3\omega)\delta a(t) + \frac{\phi_0}{2}(1+\omega)\delta\phi(t) = 0, \tag{32}$$

which gives the solution

$$\delta a(t) = C_1 e^{\lambda_3 t} + C_2 e^{-\lambda_3 t} + C_3 e^{\lambda_4 t} + C_4 e^{-\lambda_4 t}, \delta\phi = C_5 e^{\xi t}, \tag{33}$$

where the parameters λ_3, λ_4 and ξ are given as

$$\begin{aligned} \lambda_{3,4} = \Bigg[& \frac{\sigma\mu^4 + 18\rho_0(1+\omega)}{36\sigma\mu^4} [2\sigma\mu^4 - 3[3\rho_0(1+\omega) \\ & \pm \sqrt{\rho_0(1+\omega)[2\sigma\mu^4(-1+3\omega) + 9\rho_0(1+\omega)]}]]\Bigg]^{\frac{1}{2}}, \end{aligned} \tag{34}$$

$$\xi = (1+\omega), \tag{35}$$

respectively. Similarly like in the case I, we verified that for a positive cosmological constant and for $\sigma = -1$, the solutions were unstable. Considering $\lambda > 0$ and $\sigma = 1$, the stability regions are as follows:

First Stability Region

$$\textbf{2.2A} \quad \frac{2\mu^4}{9} < \rho_0 < \frac{(5+\sqrt{41})\mu^4}{12}, \tag{36}$$

$$\frac{2\mu^4 - 9\rho_0}{3(2\mu^4 + 3\rho_0)} \leqslant \omega < \frac{1}{9}\left[-6 + \sqrt{\frac{3(2\mu^4 + 3\rho_0)}{\rho_0}}\right]. \tag{37}$$

Second Stability Region

$$\textbf{2.2B} \quad \rho_0 = \frac{(5+\sqrt{41})\mu^4}{12}, \tag{38}$$

$$-\frac{7 + 3\sqrt{41}}{3(13+\sqrt{41})} < \omega < \frac{1}{3}\left[-2 + \sqrt{\frac{13+\sqrt{41}}{5+\sqrt{41}}}\right]. \tag{39}$$

Third Stability Region

$$\textbf{2.2C} \quad \rho_0 > \frac{(5+\sqrt{41})\mu^4}{12}, \tag{40}$$

$$-\frac{\mu^4 + 3\rho_0}{9\rho_0} \leqslant \omega < \frac{1}{9}\left[-6 + \sqrt{\frac{3(2\mu^4 + 3\rho_0)}{\rho_0}}\right]. \tag{41}$$

These solutions are consistent with the stability condition $f_{RR} = d^2f/dR^2 > 0$.

Figure 2: Stability Region for $f \propto R + \frac{1}{R} + \phi$

plots in figure (2) shows the areas of stability corresponding to the case II of $f(R,\phi) \propto R + \frac{1}{R} + \phi$. As before, the GR represented by thick solid line. In the bottom of region left, which approaches to GR, the Λ is negative. In the upper region shaped like triangular (2.2A, 2.2B and 2.2C), the static universe of Einstein is stable and $\Lambda \geqslant 0$ positive. This shows that the $f(R,\phi)$ gravity provides a broader and more robust framework for understanding the stability of cosmological models, especially for regions where Λ is positive.

3. Conclusion

In this work, we have investigated the stability of the Einstein static universe within the framework of $f(R, \phi)$ gravity. By introducing a scalar field ϕ alongside the Ricci scalar R, In this work, we derived the stability conditions for perturbations around the Einstein static solution by examining two distinct forms of the $f(R, \phi)$ function. Using linear perturbation theory, we identified the regions in which these solutions remain stable. Our approach began with the derivation of the field equations for $f(R, \phi)$ gravity, followed by the application of linear perturbations to the Einstein static background. In the theoretical context described above, we studied the influence of the scalar field (ϕ) on gravitational dynamics in both remote and recent epochs of the universe. The results obtained indicate that the addition of ϕ leads to a stabilzation of Einstein's static universe and makes the theoretical space for simulating cosmic acceleration more flexible. Despite this promising results, he $f(R, \phi)$ approach also poses some intrinsic obstacles. Even in this case, although the stability of the Static Einstein solution increases with the addition of a scalar field, but at the same time makes a mathematics more complex. Such an increase in complexity could also prevent the model from its realistic application to some cosmological context. The stability regions depend very much on the specific $f(R, \wp)$ and the form of coupling between matter and scalar field. These particular dependencies require additional in-depth investigation. Furthermore, in our present consideration we have concentrated more on uniform perturbations. An in-depth understanding of the model itself demands further study of inhomogeneous modes (non-uniform disturbances) which can trigger additional instabilities. As is the case of any modified gravity theory, a detailed analysis of how well the model agrees with observation—particularly those aspects connected to structure formation on large scales and late-time acceleration—is still an imperative exercise. As compared to the pure GR, $f(R, \phi)$ theory is a more effective way to realize the stability of cosmological solutions such as the static Einstein universe. In the framework of the standard framework of GR, such solutions are unstable by their nature and can survive only under very fine-tuned circumstances. On the other hand, the scalar field of $f(R, \phi)$ theory does change gravitation nature perse in such a way of making stability region larger. Moreover, in contrast to the case of the general $f(R)$ gravity considered above, the ϕ field is also accompanied by additional degree of freedom. This extra freedom also help to complement the stability domains and allow more flexibility in tuning the equations of state and their corresponding gravitational interactions, making our model to become much more physically acceptable.

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