J. Mod. Trends Math. Sci., 1 (2025), 1–18

Online: ISSN 2008-949X

JOURNAL OF MODERN TRENDS IN MATHEMATICAL SCIENCES JMTMS

Decision-Making Based on Dual Probabilistic Linguistic Term Sets with Incomplete Assessments of Comparative Linguistic Expressions

Rabiah Bibi[a,*], Badar-un-nissa[b]

[a] *Department of Mathematics, Minhaj University Lahore, Pakistan.*

[b] *Department of Mathematics, Minhaj University Lahore, Pakistan.*

Abstract

Decision making in multi-criteria group situations usually follows a qualitative process. Natural language is preferred by decision makers (DMs) when conveying information in the decision making process. It is imperative for DMs to Utilizing the Probabilistic Linguistic Term Sets (PLTS) as a means of displaying information about their respective decisions. A number of decision-making processes rely on probabilistic linguistic term sets (PLTSs) because the ease of assessment and the ability to use probability information. To improve representations of real-world complexity and uncertainty, this work is to introduce the dual probabilistic linguistic term sets (DPLTSs), which is an extended form of PLTS. When evaluating alternatives, sometimes the evaluators can only give a comparative assessment and evaluators sometimes struggle to understand all the alternatives and cannot give a comprehensive evaluation. Therefore, in this work, comparative linguistic expressions (CLEs) are transformed into Dual Probabilistic Linguistic Term Sets (DPLTSs) and alternatives are evaluated by DPLTSs in CLEs. As a result, the probability distribution of the transformation was adjusted for greater consistency with common sense. Furthermore, we identify some basic DPLTS operations laws. An accuracy function and a score function are defined in order to compare two DPLTSs. We propose that an evaluation decision making method can be developed, combining both CLEs and incomplete assessments. This work is finally accomplished by using a numerical example to illustrate the use of the proposed method. Calculation of the result proved to be simple, straightforward and e¤ective. It is possible to apply the proposed method successfully to other selection problems as well, resulting in the selection of suitable alternatives. According to the results of this study, DPLTS-based evaluations and decision-makings are more exible and less prone to limitations.

Keywords: FS, HFS, HFLTS, PLTS, PLN, DPLTS, CLE, Decision making

2020 MSC: 03E72, 03E70, 68T37.

1. Introduction

Fuzzy set theory is widely used today, originally developed by Zadeh (Zadeh 1965)v[28]. A fuzzy set A on the universe of discourse Y is a mapping from Y to $[0,1]$. It is represented by $A = \{(y, A(y)\}$. For any $y \in Y$, the value $A(y)$ is called the degree of membership of y in A i.e., $A(y)$ =Degree $(y \in A)$, and the map $A : Y \to [0,1]$ is called a membership function. To convey an endorsed setting for inadequate and standard data is the assurance of fuzzy set theory, as imparted by people in normal verbal. The possibility

*Corresponding author

Email addresses: rabiaboota786@gmail.com (Rabiah Bibi), badarunnissa9@gmail.com (Badar-un-nissa)

doi: 10.22436/...

Received: December 03, 2025 Revised: December 09, 2025 Accepted: December 12, 2025

of fuzzy set offers a typical methodology of managing issues which have ambiguity information, what's more, normally the symbolization of fuzzy set theory is totally non-factual. In 1965, after the presence of a fuzzy set hypothesis, the general assignment of looking on relations was begun by Pro Lotfi A. Zadeh itself in 1971[30].

The fuzzy connection is presented by Lotfi. A. Zadeh [29] in his developmental work, all around characterized the symbolization of resemblance as a consensus of the symbolization of likeness and offered the possibility of fuzzy requesting. (Herrera *et al.* 1995)[7] utilized the linguistic quantifier to use linguistic preference relations to express the collective preferences. After presenting the mathematical form of linguistic preference relations, Xu (2006) [25] presented a model of incomplete representation. In light of the fact that linguistic terms cannot accurately express a preference for one thing over another, combining multiple linguistic terms with probabilities may enable one to build preference relationships while also accommodating uncertainty. With the introduction of pairwise correlation and differing data, Alonso [1], Chiclana and Herrera-Viedma [1] presented a methodology to assess missing preferences. Utilizing only the preferences provided by the master in an incomplete fuzzy preference connection, the strategy aims to assess the missing data.

According to Yang and Xu [24], unassigned probability is human cognition's ignorance; thus, the ignorance should not be associated with any specific linguistic term in the linguistic term set. A good alternative for distributing the uninformed probability is to use the power set of linguistic terms. The concept of PLTSs has been introduced by Pang *et al.* (2016) [15] as an extension to traditional LTSs. PLTSs enable DMs to deliver many possible linguistic classifications and probabilistic information about alternatives/objects, while also coping with partially incomplete evaluations, contrary to previous representation approaches. The DMs may provide imperfect probabilistic information, referred to as ignorance, under the concept of PLTSs. Tradition dictates that PLTSs must be normalized as a way of diminishing ignorance. In some cases, however, this isn't possible. In addition to losing information, normalization can also result in inaccurate results. The methods of operation and comparison were developed by Pang *et al.*(2016). The PLTS comparison has been studied extensively during the decision-making process.

In a study conducted by Pang *et al.*[14] and Wu *et al.* [23], a comparison method using scores and variances was proposed. Based on the probability degree, Chen [3], Bai [2] and Feng [5] showed how to compare PLEs. Making a decision, they also do not examine whether PLTSs are similar or whether complex expressions are transformed. PLTSs can be ranked using a methodology created by Pang *et al.* (2016) [13]. In reviewing Pang's approach to comparison, we note that the degree of deviations from the corresponding PLTS score or score is taken into account. His variation functions were developed so that PLTSs could be compared using the score function to distinguish the equal scores of two PLTSs. In addition, he proposed a method of considering the disregarded probability proportionally based on existing language concepts.

Chinese researchers Gao *et al.* [6] and Wu and Liao [20] investigated probabilistic linguistic preference relations (PLPRs), which examine how alternatives relate to one another and how to take decisions with incomplete PRPLs. It is also examined whether PLPR has consensus and consistency. As this method relies on comparisons, it is necessary to know the comparisons between each other. Decision makers' preferences as measured by their distance from each other, according to Zhang *et al.* (2016) [27], serves as the consensus measure for PLPRs and the minimum consensus values, it is important to consider pairwise decision-makers a consensus measure for group decision-makers. Zhao *et al.* (2018) define the consensus measure as one minus PLTS similarity degrees. According to the probability degrees of PLTSs, similarity degrees between PLTSs were calculated. For large-scale group decision-making, he proposed a consensus measure based on determining sub-group consensus separately. A description of probabilistic-based expressions to support qualitative decision making was presented by Xu *et al.* (2018) [26]. He focused on probabilities-based expressions and introduced roughly three decision-making approaches and their associated applications, along with the comparison rules, measurements, and aggregation operators.

PLTS provides humans with a natural means to express themselves as well as providing an imprecise assessment of humans in the decision-making processes. It is highly effective for aggregating ideas between different people denoted by linguistic terms. Aggregation functions are very useful not only for displaying the overall performance across multiple criteria, but also for characterizing them. Pang *et al.* (2016) [13] propose four aggregation types as the probabilistic linguistic averaging, weighing, geometric and weighing geometric. A method was presented by Rodriguez *et al*. [16] for transferring CLE (comparative linguistic expression) into HFLTS using context-free grammars. Alvaro Labella et al.[10] suggested a CLE that incorporated symbolic translation into PLTS called the extended comparative linguistic expressions with symbolic translation. A representation model of CLE was proposed by Liu *et al [12].* based on the inherent vagueness and uncertainty in CLE: the type-2 fuzzy envelope for hesitant fuzzy linguistic terms. Therefore, the model reduces the complexity of CLE calculations, thus increasing the effectiveness of information and eliminating the loss of information as a result of transformation. Durand and Truck (2018) [4] focused on qualifiers and modifiers for linguistic expressions of hesitations to determine the probability, also known as the weight of linguistic terms, for hesitant verbal expressions on preference assessments. When linguistic terms are used in linguistic statements, the probabilities of their use are calculated. Six qualifiers and four modifiers were proposed. Six qualifiers can be used to describe hesitations: "between... and; ... or..., at most... or less than...; at least... or more than...; everything except..." Four modifiers from linguistic assertions, such as "truly, rather, a little bit, tend to lean toward," can represent hesitations, two are reinforcing, and the other two are weakening. We can calculate linguistic probabilities by using qualifiers and modifiers and weight-assigning methods for qualifiers and modifiers. PLTS differs significantly from other linguistic expression models by constraining the sum of probabilities.

The models proposed by Durand and Truck [4], the sum of the probabilities associated with linguistic terms should be equal to 1. A more general restriction than the constraints of the previous models pertains to the PLTS, in which the sum is not greater than 1. The probabilities of linguistic terms were determined by Durand and Truck (2018) [4] with a superior set of rules than Pang *et al.* (2016) [15]. The probabilistic uncertain LTS presented by Lin *et al.* (2018) defines linguistic intervals of specific PLTSs, while the interval-valued PLTS offered by Bai *et al.* (2018) [2] converts crisp probabilities to interval probabilities. (Jin *et al.* 2019) [8] took into account upper and lower bounds for probability restriction in their uncertain performance metrics. This differs from interval-valued PLTS. The PLTS normalization method used by Zhang and Xing differs from that used by Zhang *et al*. [27]. In this method, the ignorant probability is assigned to the whole linguistic term set. Therefore, the ideal solution can contain distinct linguistic terms, making it more flexible than Zhang *et al.* (2019). A second method to obtain the ideal solutions without normalization is also presented by Wang *et al.* [21]. The full LTS $\{\phi\}$, as defined by Liao *et al*. (2019) [11] can also be assigned the missing probabilities. Still, the original information is retained from the DM. Based on three distinct attitudes of decision-makers, Song and Li [17] assigned the ignorant probability using three distinct principles. For the most important, average, and smallest linguistic terms of the PLTS, the ignorant probability should be assigned according to the optimistic, neutral, and pessimistic attitudes. In the emergency decision making process, Gao *et al.* (2019) [6] analyzed the incomplete PLPR. With the fault tree analysis, it is possible to estimate the missing elements of an incomplete PLPR.A PLPR is incomplete if there are missing elements. Krishnakumar *et al.* (2019) [9] calculated the missing elements using the additive consistency property of PLPR. A consensus measure is calculated by summing the projection between collective PLPRs and individual PLPRs, as given by Liu et al. (2019).

As part of their new work on decision-making sets, Xie et al. (2017) [23] introduced the concept of DPLTSs, which can accommodate information on two aspects of a word that are relevant from both perspectives of membership degrees and nonmembership degrees. Similar features characterize DPLPRs that are based on DPLTSs. Furthermore, due to a wide range of subjective and objective reasons and the absence of other preference information , the DPLPRs cannot be determined adequately using incomplete preference information in multicriteria decision-making. Since this information is incomplete, it can be challenging to determine the weight of the criteria. Our goal is to solve a decision-making problem with unknown weights for multiple criteria. In the context of dual probabilistic linguistics situation, incomplete PRs fall into two categories: the linguistic category and the probabilistic category. However, despite the fact that all steps leading to a perfect DPLPR are consistent, it is not possible to guarantee the total consistency of these DPLPRs [23]. Due to this, the consistency of the DPLPRs obtained must be evaluated as much as possible in order to avoid making unreasonable decisions. In a DPLTS, the membership part is associated with the non-membership part, which provides decision-making information. Both the membership and non-membership elements are made up of many linguistic terms and the associated probabilities. Consequently, the DPLTS can provide as much information about the decisions as possible.

In this study, linguistic multicriteria decision-making under incomplete probabilistic conditions is investigated. The differences can be explained by two factors: first, the experimental data scale is too small. Another difference is that the information used to make decisions takes different forms. With dual probabilistic linguistic decision-making information, it becomes possible to infer the degree of liking and disliking of things being evaluated by the evaluator. When choosing the most appropriate method, it is imperative to use all the available preferences information. Therefore, dual probabilistic linguistic term sets are more suitable for collecting preference data than probabilistic linguistic term sets. The objectives of the study is given as follows:

1. To establish a method of transforming the comparative linguistic expressions with incomplete evaluation based on dual probabilistic linguistic terms.
2. To develop a method for correcting deviations caused by incomplete assessments.
3. To rank a given set of alternatives in the presence of different criteria with DPLTSs.
4. To solve the practical applications of the proposed methodology using real-life problems.

The rest of paper is structured as follows. The foundational concepts like FS, HFS, HFLTS, PLTS, PLN, DPLTS and CLE are provided in Section 2. Section 3 provides first the method of transformation of CLEs into DPLTS with incomplete assessments from the DMs and then provide a technique to rank aternatives under given criteria. Section 4 illustrates a numerical example showing the effectiveness of the proposed study. Section 5 draws a conclusion and reflects on the most important findings and implications of future research.

2. Main Results

In this section, few basic definitions are provided that can help to recall the knowledge related to the topic of propsed study.

Definition 2.1. [28] The fuzzy set A represents a mapping from Y to $[0,1]$. It is represented by $A = \{(y, A(y)\}$. For any $y \in Y$, the value $A(y)$ is called the degree of membership of y in i.e., $A(y)$ = Degree $(x \in A)$, and the map $A : Y \to [0,1]$ is called a membership function. $F(Y)$ represents the class of every single fuzzy set in Y

Definition 2.2. [28] The membership function charts every component of fuzzy sets to $[0,1]$. $\lambda_A : Y \to [0,1]$ Where $[0,1]$ depicts real numbers between 0 and 1 (covering 0and 1). Therefore, fuzzy sets are more vague than crisp sets. As a subset of universal set, crisp set is reflected. In a similar way, Set A of fuzzy sets can be seen as a subset of set Y of universal sets.

Definition 2.3. [18] Let Y be a fixed set while E is a hesitant fuzzy set (HFS) on Y which, with respect to Y, returns a subset of $[0,1]$.To make HFSs more understandable, Xia and Xu (2011) presented them in mathematical symbols:

$$E = \{\langle y, h_E(y)\rangle \mid y \in Y\},$$

$h_E(y)$ denotes the degrees of membership to which the element $y \in Y$ may belong to the set E, the values in $[0,1]$. The hesitant fuzzy element (HFE) is referred to as $h_E\ (y)$.

The concept of HFLTS [22], which is a combination of fuzzy linguistics and HFS, is presented as an extension of the HFS concept.

Definition 2.4. [22] Let Y be a fixed set and $s_g(y) \in S$. An HFLTS B in Y is an object:

$$B = \{\langle y, s_g(y), h_B(y)\rangle \mid y \in Y\}$$

where $h_B(y)$ is a set of finite numbers in $[0,1]$and indicate the degree to which y belongs to $s_g(y)$. When, $Y = \{y_1, y_2, ...y_n\}$ has only one element, HFLS B is reduced to

$$\langle s_g(y), h_B(y)\rangle$$

For computational convenience,

$$\alpha = \langle s_g(\alpha), h_B(\alpha)\rangle$$

is called an HFLN. When $h_B(y) = \{r\}$ has only one element, it indicates that the degree that y belongs to $s_g(y)$ is r. For example, $\langle s_4, 0.5\rangle$is called a fuzzy linguistic number, which is a special case of HFLNs.

Definition 2.5. [15] Assume $S = \{s_g | g = -\pi, ..., -1, 0, 1, ..., \pi\}$ is a LTS. A PLTS would be as follows:

$$M(p) = \left\{ M^{(k)}(p^{(k)}) \mid M^{(k)} \in S,\ r^{(k)} \in g,\ \ p^{(k)} \geqslant 0, k = 1, 2, ..., \#M(p),\ \sum_{k=1}^{\#M(p)} p\ (k) \leqslant 1 \right\}$$

Where $M(p)$ represents a possible membership degree and $M^{(k)}(p^{(k)})$is the linguistic term $M^{(k)}$ involved in the probability $p^{(k)}$, $r^{(k)}$ is the subscript of $M^{(k)}$ and $\#M(p)$ the total number of linguistic terms in $M(p)$.

Definition 2.6. [15] Let

$$M_1(p) = \left\{ M_1^{(k)}(p_1^{(k)}) \mid k = 1, 2, ..., \#M_1(p) \right\}$$

and

$$M_2(p) = \left\{ M_2^{(k)}(p_2^{(k)}) \mid k = 1, 2, ..., \#M_2(p) \right\}$$

be two PLTSs must be defined as above, so

1. $M_1(p) \oplus M_2(p) = \oplus_{\ M_1^{(k)} \in M_1(p), M_2^{(k)} \in M_2(p)} \{p_1^{(k)} M_1^{(k)} \oplus p_2^{(k)} M_2^{(k)}\}$

2. $\mu M(p) = \oplus_{\ M^{(k)} \in M(p)} \{\mu p_1^{(k)} M_1^{(k)}\}, \quad \mu \geqslant 0$

3. $\mu_1 M_1(p) \oplus \mu_2 M_2(p) = \oplus_{\ M_1^{(k)} \in M_1(p), M_2^{(k)} \in M_2(p)} \{\mu_1 p_1^{(k)} M_1^{(k)} \oplus \mu_2 p_2^{(k)} M_2^{(k)}\},\ \mu_1,\ \mu_2 \geqslant 0$

Definition 2.7. [15] When given a PLTS $M(p)$ with $\sum_{k=1}^{\#M(p)} p\ (k) < 1$ The associated normalized PLTS $M^*(p)$ is defined as follows

$$M^*(p) = \left[M^{(k)}(p^{*(k)}) \mid k = 1, 2, ..., \#M(p) \right],$$

where $p^{*(k)} = \frac{p(k)}{\sum_{k=1}^{\#M(p)} p(k)}$ for all $k = 1, 2, ..., \#M(p)$,

Example 2.8. Let $M_1(p) = \{s_3(0.2), s_4(0.7)\}$ be a PLTS. The normalized PLTS $M^*(p) = \{s_3(0.23), s_4(0.77)\}$ where $p^{*(1)} = \frac{0.2}{0.2+0.7} = 0.23$,$p^{*(2)} = \frac{0.7}{0.2+0.7} = 0.77$

The PLWA operator was proposed in order to aggregate information from PLTS's with different weights.

Definition 2.9. Let

$$M_i(p) = \left[M_i^{(k)}(p_i^{(k)}) \mid k = 1, 2, ..., \#M_i(p)\right] \; (i = 1, 2, ..., N)$$

be N PLTSs, where $M_i^{(k)}$and $p_i^{(k)}$ represent the kth linguistic term and its probability respectively in $M_i(p)$. Then

$$PLWA(M_1(p), M_2(p), ..., M_N(p)) = w_1 M_1(p) \oplus w_2 M_2(p) \oplus, ..., \oplus w_N M_N(p)$$

$$= \cup_{M_1^{(k)} \in M_i(p)} \{w_1 p_1^{(k)} M_1^{(k)}\} \oplus \cup_{M_2^{(k)} \in M_2(p)} \{w_2 p_2^{(k)} M_2^{(k)}\} \oplus, ..., \oplus \cup_{M_N^{(k)} \in M_N(p)} \{w_N P_N^{(k)} M_N^{(k)}\}$$

it is called the PLWA operator, where$w = (w_1, w_2, ..., w_N)^T$, $wi \geqslant 0, i = 1, 2, ...N$, and $\sum_{i=1}^{N} w_i = 1$ is the weight of $M_i(p)$.

Definition 2.10. [23] A DPLTS on Y, when Y is a fixed set, is defined as follows:

$$D(p) = [\langle y, M(p), N(p) \rangle, y \in Y]$$

Where,

$$M(p) = \left\{ M^{(k)}(p^{(k)}) \mid M^{(k)} \in S, p^{(k)} \geqslant 0, k = 1, 2, ..., \#M(p), \sum_{k=1}^{\#M(p)} p\,(k) \leqslant 1 \right\},$$

$$N(p) = \left\{ N^{(l)}(p^{(l)}) \mid N^{(l)} \in S, p^{(l)} \geqslant 0, l = 1, 2, ..., \#N(p), \sum_{l=1}^{\#N(p)} p\,(l) \leqslant 1 \right\}$$

$M(p)$ and $N(p)$ indicate the possibility of membership and non-membership for $y \in Y$. Moreover,$s_{-\iota} \leqslant M^+ \oplus N^+ \leqslant s_\iota, s_{-\iota} \leqslant M^- \oplus N^- \leqslant s_\iota$ where, $M^+$and M^- refer to the maximum and minimum elements of the PLTS,$M(p)$, respectively and N^+ and N^- are the maximum and minimum elements of the PLTS, $N(p)$. In addition, we call the pair $D(p) = \langle M(p), N(p) \rangle$ the dual probabilistic linguistic element (DPLE).

Note: If $M(p) = \phi$ and $N(p) = \phi$, The DPLTS then becomes the PLTS.

Definition 2.11. [23] Let $S = \{s_g | g = -\pi, ..., -1, 0, 1, ..., \pi\}$ be a LTS and $D(p) = \langle M(p), N(p) \rangle$, $D_1(p) = \langle M_1(p), N_1(p) \rangle$ and $D_2(p) = \langle M_2(p), N_2(p) \rangle$ be three DPLEs where,

$$M_i(p) = \left\{ M_i^{(k)}(p_i^{(k)}) \mid M_i^{(k)} \in S,\; p_i^{(k)} \geqslant 0, k = 1, 2, ..., \#M_i(p_i), \sum_{k=1}^{\#M_i(p_i)} p_i^{(k)} \leqslant 1 \right\}$$

$$N_i(p) = \left\{ N_i^{(l)}(p_i^{(l)}) \mid N_i^{(l)} \in S,\; p_i^{(l)} \geqslant 0, l = 1, 2, ..., \#N_i(p_i), \sum_{l=1}^{\#N_i(p_j)} p_i^{(l)} \leqslant 1 \right\}$$

and μ must be a positive real number. Where$i = 1, 2$; h and h^{-1} be two reversible transformation functions for possible membership degrees $M_i(p)$ and g and g^{-1} be two reversible transformation functions for non- membership degrees $N_i(p)$. By combining the linguistic variable s_g and the membership degree λ, you can realize the inter-transformation of both variables.

$$h : [-\pi, \pi] \longrightarrow [0, 1]\,, h(M(p)) \; = \{[\frac{r^{(k)}}{2\pi} + \frac{1}{2}](p^k)\} = M_\lambda(p),\; \lambda \in [0, 1]\,;$$

$$h^{-1} : [0, 1] \longrightarrow [-\pi, \pi]\,, h^{-1}(M_\lambda(p)) = \{S_{(2\lambda-1)\pi}(p^\lambda) \mid \; \lambda \in [0, 1]\} = M(p)$$

and

$$g:[-\pi,\pi]\longrightarrow[0,1],h(N(p))\ =\{[\frac{r^{(l)}}{2\pi}+\frac{1}{2}](p^{l})\}=N_{\lambda}(p),\ \lambda\in[0,1];$$

$$g^{-1}:[0,1]\longrightarrow[-\pi,\pi],h^{-1}(N_{\lambda}(p))=\{S_{(2\lambda-1)\pi}(p^{\lambda})\mid\ \lambda\in[0,1]\}=N(p)$$

Then

1. $D_1\oplus D_2=\bar{U}_{M_1\in M_1^{(K)},\,p_{M_1}\in p_1^{(K)},\,N_1\in N_1^{(l)},\,p_{N_1}\in p_1^{(l)},\,M_2\in M_2^{(K)},\,p_{M_2}\in p_2^{(K)},\,N_2\in N_2^{(l)},\,p_{N_2}\in p_2^{(l)}}$

   $\{g^{-1}\{(g(M_1)+g(M_2)-g(M_1)g(M_2))(p_{M_1}p_{M_2})\},\{(N_1\otimes N_2)(p_{N_1}\ p_{N_2})\}\}$
2. $D_1\otimes D_2=\bar{U}_{M_1\in M_1^{(K)},\,p_{M_1}\in p_1^{(K)},\,N_1\in N_1^{(l)},p_{N_1}\in p_1^{(l)},\,M_2\in M_2^{(K)},\,p_{M_2}\in p_2^{(K)},\,N_2\in N_2^{(l)},\,p_{N_2}\in p_2^{(l)}}$

   $\{\{(M_1\otimes M_2)(p_{M_1}p_{M_2})\},\{g^{-1}\{(g(N_1)+g(N_2)-g(N_1)g(N_2))(p_{N_1}\ p_{N_2})\}\}$
3. $\mu D\ =\bar{U}_{\ M^{(K)}\in M(p),\,N^{(l)}\in N(p)}\{\{g^{-1}(1-(1-g(M^{(K)}))^{\mu}(p^{(K)})\},\{(N^{(l)})^{\mu}(p^{(l)})\}\}$
4. $D^{\mu}=\bar{U}_{\ M^{(K)}\in M(p),\,N^{(l)}\in N(p)}\{(M^{(k)})^{\mu}(p^{(k)})\},\{g^{-1}(1-(1-g(N^{(l)}))^{\mu}(p^{(l)})\}$ where $\mu\geqslant 0$

An elementary aggregation operator for the DPLTSs be used to integrate DPLEs more effectively in the practical decision-making process:

Definition 2.12. [23] Let $D_i(p)=\langle M_i(p),N_i(p)\rangle(i=1,2,...,n)$ be a DPLE, Then

$$DPLWA(D_1,D_2,...,D_n)=\oplus_{i=1}^{n}w_iD_i$$

Where w_i is the weight vector of $D_i(p)=\langle M_i(p),N_i(p)\rangle$, $w_i\geqslant 0, i=1,2,...,n$; and $\sum_{i=1}^{n}w_i=1$.

Definition 2.13. [23] Let $D(p)=\langle M(p),N(p)\rangle$ be a DPLE, then the score function of the DPLE is

$$E\,(D)=S_{\varphi\text{-}\phi}$$

Where,

$$\varphi=\frac{\sum_{k=1}^{\#M(p)}r^{(k)}p^{(k)}}{\sum_{k=1}^{\#M(p)}p^{(k)}},\qquad \phi=\frac{\sum_{l=1}^{\#N(p)}r^{(l)}p^{(l)}}{\sum_{l=1}^{\#N(p)}p^{(l)}}$$

For two DPLEs $D_i\ (i=1,2)$ if $E\,(D_1)>E\,(D_2)$, the D_1 is superior to D_2, denoted as $D_1\succ D_2$;if $E\,(D_1)<E\,(D_2)$, then D_1 is inferior to D_2, denoted as $D_1\prec D_2$. If $E\,(D_1)=E\,(D_2)$, When two DPLTEs occur at the same time, it is hard to distinguish them. So, we can define the accuracy function for DPLE as follows:

Definition 2.14. [23] Let $D(p)=\langle M(p),N(p)\rangle$ be a DPLE, then DPLE accuracy functions are as follows:

$$\alpha\,(D)=\frac{\left[\sum_{k=1}^{\#M(p)}\left(p^{(k)}\left(r^{(k)}\text{-}\ \varphi\right)\right)^{2}\right]^{\frac{1}{2}}}{\sum_{k=1}^{\#M(p)}p^{(k)}}+\frac{\left[\sum_{l=1}^{\#N(p)}\left(p^{(l)}\left(r^{(l)}\text{-}\ \phi\right)\right)^{2}\right]^{\frac{1}{2}}}{\sum_{l=1}^{\#N(p)}p^{(l)}}$$

If $\alpha\,(D_1)>\alpha\,(D_2)$ then $D_1\succ D_2$; if $\alpha\,(D_1)<\alpha\,(D_2)$ then $D_1\prec D_2$ and if $\alpha\,(D_1)=\alpha\,(D_2)$ then $D_1\sim D_2$ for two DPLEs $D_i\ (i=1,2)$ with $E\,(D_1)=E\,(D_2)$.

3. Transformation of CLEs into DPLTS with Incomplete Assessments for Ranking Alternatives

The CLEs are developed using HFLTS and using context-free grammars as similar to what experts do when making decisions in uncertainty group decision situations even when they become stuck between different terms in which decision makers must elicit preferences. It is possible for experts to hesitate between different linguistic terms when expressing their preferences in decision-making situations with high level of uncertainty. The method context-free grammar is similar to what humans do cognitively and is useful for generating comparative linguistic expressions. It was demonstrated by Rodriguez et al. [16] that context-free grammar allows for the production of comparative linguistic expressions.

Definition 3.1. Suppose that $E_{\acute{g}_h}$ is a function that converted HFLTS H_S from linguistic expressions accessed by $\acute{g}_h$. The context-free grammar $\acute{g}_h$ uses the linguistic term set S . H_S is a complement to HFLTS, and $\acute{g}_h$ is the grammar for context-free grammar. Here is how function $E_{\acute{g}_h}$ is defined:

1. $E_{\acute{g}_h}(s_i) = \{s_i | s_i \in S\}$
2. $E_{\acute{g}_h}(\text{less than } s_i) = \{ s_j | s_j \in S \text{ and } s_j \leqslant s_i\}$
3. $E_{\acute{g}_h}(\text{greater than } s_i) = \{ s_j | s_j \in S \text{ and } s_j \geqslant s_i\}$
4. $E_{\acute{g}_h}(\text{between } s_i \text{ and } s_j) = \{ s_k | s_k \in S \text{ and } s_i \leqslant s_k \leqslant s_j\}$

Thus, context-free grammar expressions can be converted into HFLTS for example, let S = {none, fair, very fair, average ,good, very good, excellent} be a linguistic term set where $E_{\acute{g}_h}$ can be defined as follows:

1. $E_{\acute{g}_h}$(fair)= {fair}
2. $E_{\acute{g}_h}$(lower than good) ={ none, fair, very fair, average ,good }
3. $E_{\acute{g}_h}$(greater than good) = { good, very good, excellent }
4. $E_{\acute{g}_h}$(between average and excellent) = { average ,good, very good, excellent }

Definition 3.2. Let $\acute{g}_h$ be the grammar which combines DPLTSs and CLEs to assist experts in expressing with more flexibility, while evaluators' CLEs are denoted by cl.

Membership degrees may take the following forms:

1. cl_1 = lower than M_1
2. cl_2 = greater than M_1
3. cl_3 = lower than M_1 and M_2
4. cl_4 = greater than M_1 and M_2
5. cl_5 = between $M_1$and M_2

Where M_1 and M_2 are the membership linguistic terms in DPLTSs. Similarly, non-membership degrees may take the following forms:

1. cl_1 = lower than N_1
2. cl_2= greater than N_1
3. cl_3 = lower than N_1 and N_2

4. cl_4 = greater than N_1 and N_2

5. cl_5 = between N_1 and N_2

Where N_1 and N_2 are the non-membership linguistic terms in DPLTSs.

CLEs with incomplete assessments can be used to make decisions in DPLTSs environment. The following key steps can be defined as follows:

STEP 1: Correct Information

The reference domain should be determined after setting ϕ and φ with respect to the number of evaluators available and alternatives under certain criteria. It will also depends upton the nature of the problem. Use this method to adjust the evaluation afterward. Evaluators $E_1, E_2, ..., E_N$ use DPLTSs to Identify alternatives $B_1, B_2, ..., B_M$ and $S = \{s_g | g = -\pi, ..., -1, 0, 1, ..., \pi\}$. Assuming Y is a fixed set, a DPLTS on Y would be:

$D_{ij}(p) = \left[\langle y, M_{ij}(p), N_{ij}(p)\rangle, y \in Y\right]$

Where,

$M_{ij}(p) =$

$\left\{M_{ij}^{(k)}(p_{ij}^{(k)}) \mid M_{ij}^{(k)} \in S, r_{ij}^{(k)} \in g, p_{ij}^{(k)} \geqslant 0, k = 1, 2, ..., \#M_{ij}(p_{ij}), \sum_{k=1}^{\#M_{ij}(p_{ij})} p_{ij}\,(k) \leqslant 1\right\}$

and

$N_{ij}(p) =$

$\left\{N_{ij}^{(l)}(p_{ij}^{(l)}) \mid N_{ij}^{(l)} \in S, q_{ij}^{(l)} \in g, p_{ij}^{(l)} \geqslant 0, l = 1, 2, ..., \#N_{ij}(p_{ij}), \sum_{l=1}^{\#N_{ij}(p_{ij})} p_{ij}\,(l) \leqslant 1\right\}$

$M_{ij}(p)$ and $N_{ij}(p)$ indicate the degree to which $y \in Y$ can be a member and non-member of that element is the normalized DPLTS evaluated by the ith evaluator of the jth alternative ($i \in [1, N]$,and $i \in Z^+; j \in [1, M]$, and $j \in Z^+$), where $r_{ij}^{(k)}$ and $q_{ij}^{(l)}$ is the subscript of $M_{ij}^{(k)}$ and $N_{ij}^{(l)}$ respectively, and $\#M_{ij}(p)$ and $\#N_{ij}(p)$ are the number of linguistic terms in $M_{ij}(p)$ and $N_{ij}(p)$ respectively. Select D_α and D_β respectively as the reference domains for the evaluators evaluations of some alternatives. The alternative must be evaluated by more than γ of the evaluators in the reference domain. Furthermore, the evaluator should consider a broad range of alternatives, where , ϕ and $\varphi \in [0, 1]$. The reference domain must also contain expressions with DPLTSs without any CLEs. In our view, an evaluator's average score reflects their scoring standard in a given domain. Therefore,

$D_{ij}(p) = \left[\langle y, M_{ij}(p), N_{ij}(p)\rangle, y \in Y\right]$,

Where

$M_{ij}(p) =$

$\left\{M_{ij}^{(k)}(p_{ij}^{(k)}) \mid M_{ij}^{(k)} \in S, r_{ij}^{(k)} \in g, p_{ij}^{(k)} \geqslant 0,\ k = 1, 2, ..., \#M_{ij}(p_{ij}), \sum_{k=1}^{\#M_{ij}(p_{ij})} p_{ij}\,(k) \leqslant 1\right\}$

and

$N_{ij}(p) =$

$\left\{N_{ij}^{(l)}(p_{ij}^{(l)}) \mid N_{ij}^{(l)} \in S, q_{ij}^{(l)} \in g, p_{ij}^{(l)} \geqslant 0, k = 1, 2, ..., \#N_{ij}(p_{ij}), \sum_{l=1}^{\#N_{ij}(p_{ij})} p_{ij}\,(l) \leqslant 1\right\}$

The score function of DPLTSs is calculated as follows:

$SC\left[\langle M_{ij}(p), N_{ij}(p)\rangle\right] = \langle \{s_{r_{ij}}(p_{ij}^{(1)}), s_{r_{ij+1}}(p_{ij}^{(2)})\}, \{s_{q_{ij}}(p_{ij}^{(1)}), s_{q_{ij+1}}(p_{ij}^{(2)})\}\rangle$

Subject to

$\left(\sum_{k=1}^{\#M_{ij}(p)} r^{(k)} p^{(k)} - \left(r_{ij}\ .\ p_{ij}^{(1)} -\ (r_{ij} + 1)\ .\ p_{ij}^{(2)}\right)\right) = D_{\alpha_i} - D_\alpha,\ \ p_{ij}^{(1)} + p_{ij}^{(2)} = 1$

and

$\left(\sum_{l=1}^{\#N_{ij}(p)} q^{(l)} p^{(l)} - \left(q_{ij}\ .\ p_{ij}^{(1)} -\ (q_{ij} + 1)\ .\ p_{ij}^{(2)}\right)\right) = D_{\beta_i} - D,\ \ p_{ij}^{(1)} + p_{ij}^{(2)} = 1$

where D_α and D_β is the value of DPLTS in the reference domain D, D_{α_i} and evaluator i and j provide D_{β_i} as values for the alternatives within the reference domain. After that, $D_{ij}(p)$ can be calculated using the corrected DPLTS. Thus, the SC function eliminates the effects caused by different standards of scoring.

STEP 2: To transform CLEs into DPLTSs when they are found in evaluation results when D_S represents the grammar using standard DPLTS, and $E_{\acute{g}_h}$ represents an algorithm for transforming CLEs cl, which can be obtained according to $\acute{g}_h$ into DPLTSs D_S , i.e. $E_{\acute{g}_h} : cl \longrightarrow D_S$ where $E_{\acute{g}_h}$ is set in the comparison linguistic expression as follows:

1. $E_{\acute{g}_h}(\text{lower than } D_1)$

$= \{ \sum_{i=1}^{\#M_1(p)} p_1^{(i)}.E_{\acute{g}_h}(\text{lower than } M_1^{(i)})\}, \{ \sum_{u=1}^{\#N_1(p)} p_1^{(u)}.E_{\acute{g}_h}(\text{lower than } N_1^{(u)})\}$

2. $E_{\acute{g}_p}(\text{greater than} D_1)$

$= \{ \sum_{i=1}^{\#M_1(p)} p_1^{(i)}.E_{\acute{g}_h}(\text{greater than } M_1^{(i)})\}, \{ \sum_{u=1}^{\#N_1(p)} p_1^{(u)}.E_{\acute{g}_h}(\text{greater than } N_1^{(u)})\}$

3. $E_{\acute{g}_h}(\text{lower than } D_1 \text{ and} D_2)$

$= \{ \sum_{i=1}^{\#M_1(p)} \sum_{j=1}^{\#M_2(p)} p_1^{(i)}p_2^{(j)}.E_{\acute{g}_h}(\text{lower than } \min(M_1^{(i)}, M_2^{(j)}))\},$
$\{ \sum_{u=1}^{\#N_1(p)} \sum_{v=1}^{\#N_2(p)} p_1^{(u)}p_2^{(v)}.E_{\acute{g}_h}(\text{lower than } \min(N_1^{(u)}, N_2^{(v)}))\}$

4. $E_{\acute{g}_h}(\text{greater than } D_1 \text{ and } D_2)$

$= \{ \sum_{i=1}^{\#M_1(p)} \sum_{j=1}^{\#M_2(p)} p_1^{(i)}p_2^{(j)}.E_{\acute{g}_h}(\text{greater than } \max(M_1^{(i)}, M_2^{(j)}))\},$
$\{ \sum_{u=1}^{\#N_1(p)} \sum_{v=1}^{\#N_2(p)} p_1^{(u)}p_2^{(v)}.E_{\acute{g}_h}(\text{greater than } \max(N_1^{(u)}, N_2^{(v)}))\}$

5. $E_{\acute{g}_h}(\text{between } D_1 \text{ and } D_2)$

$= \{ \sum_{i=1}^{\#M_1(p)} \sum_{j=1}^{\#M_2(p)} p_1^{(i)}p_2^{(j)}.E_{\acute{g}_h}(\text{between } M_1^{(i)} \text{and } M_2^{(j)})\},$
$\{ \sum_{u=1}^{\#N_1(p)} \sum_{v=1}^{\#N_2(p)} p_1^{(u)}p_2^{(v)}.E_{\acute{g}_h}(\text{between } N_1^{(u)} \text{and } N_2^{(v)})\}$

Where $p_1^{(i)}$ and $M_1^{(i)}$are the ith probability with respect to the linguistic term corresponding to M_1. Similarly $p_2^{(j)}$ and $M_2^{(j)}$are the jth probability with respect to the linguistic term corresponding to M_2. It should be noted that $p_1^{(u)}$ and $N_1^{(u)}$are the uth probability with respect to the linguistic term corresponding to N_1 and $p_2^{(v)}$, $N_2^{(v)}$are the vth probability with respect to the linguistic term corresponding to N_2. A linguistic expression can be turned into HFLTS using the $E_{\acute{g}_h}$ function. And then use this method to modify it.

Let $D_E(p) = E_{\acute{g}_p}(cl) = \langle\{s_{r_E^{(1)}}(p_E^{(1)}), ..., s_{r_E^{(u)}}(p_E^{(u)}), ..., s_{r_E^{(U)}}(p_E^{(U)})\},$
$\{s_{r_E^{(1)}}(p_E^{(1)}), ..., s_{r_E^{(w)}}(p_E^{(w)}), ..., s_{r_E^{(W)}}(p_E^{(W)})\}\rangle$

be a DPLTS transformed using CLE as described above. $D^i(p) = \langle\{s_{r_i^{(1)}}(p_i^{(1)}), ..., s_{r_i^{(v)}}(p_i^{(v)}), ..., s_{r_i^{(V)}}(p_i^{(V)})\},$
$\{s_{r_i^{(1)}}(p_i^{(1)}), ..., s_{r_i^{(y)}}(p_i^{(y)}), ..., s_{r_i^{(Y)}}(p_i^{(Y)})\}\rangle,$

for $i = 1, 2, ..., N$ be an object has N known evaluations. Operators of PAs are described as follows:

$PA(s_{r_E^{(u)}}, s_{r_i^{(v)}}) = \frac{\eta\left(r_E^{(u)}\right)}{\sum_{u=1}^{U}\eta\left(r_E^{(u)}\right)}, \quad PA(s_{r_E^{(w)}}, s_{r_i^{(y)}}) = \frac{\eta\left(r_E^{(w)}\right)}{\sum_{w=1}^{W}\eta\left(r_E^{(w)}\right)}$

It shows the effect on the probability of $s_{r_E^{(u)}}$ if we evaluate $s_{r_i^{(v)}}$ for the same object and it is known also. Similarly the probability of $s_{r_E^{(w)}}$ if we evaluate $s_{r_i^{(y)}}$ for the same object and it is known also. Now

$\eta(r_E^{(u)}, r_i^{(v)}) = -k.(| r_E^{(u)} - r_i^{(v)} |) + max\{r_E^{(u)} - r_E^{(1)}, r_E^{(U)} - r_E^{(u)}\}) + b,$

Where $u = 1, 2, ..., U; k, b > 0;$

$\eta(r_E^{(w)}, r_i^{(y)}) \; = -k.(|\, r_E^{(w)} - r_i^{(y)} \,|) + max\{r_E^{(w)} - r_E^{(1)}, r_E^{(W)} - r_E^{(w)}\}) + b,$

Where $w = 1, 2, ..., W; k, b > 0;$

DPLTS is calculated based on probability adjusted data

$D_{AP}(p) = \langle \{s_{r_{AP}^{(1)}}(p_{AP}^{(1)}), ..., s_{r_{AP}^{(u)}}(p_{AP}^{(u)}), ..., s_{r_{AP}^{(U)}}(p_{AP}^{(U)})\},$

$\{s_{r_{AP}^{(1)}}(p_{AP}^{(1)}), ..., s_{r_{AP}^{(w)}}(p_{AP}^{(w)}), ..., s_{r_{AP}^{(W)}}(p_{AP}^{(W)})\}\rangle$

Where,

$p_{AP}^{(u)} = \frac{1}{N}\sum_{i=1}^{N}\sum_{v=1}^{V} p_i^{(v)}.PA(s_{r_E^{(u)}}, s_{r_i^{(v)}}).p_E^{(u)}$

and

$p_{AP}^{(w)} = \frac{1}{N}\sum_{i=1}^{N}\sum_{y=1}^{Y} p_i^{(y)}.PA(s_{r_E^{(w)}}, s_{r_i^{(y)}}).p_E^{(w)}$

The adjusted DPLTS

$\langle \{s_{r_{AP}^{(1)}}(p_{AP}^{(1)}), ..., s_{r_{AP}^{(u)}}(p_{AP}^{(u)}), ..., s_{r_{AP}^{(U)}}(p_{AP}^{(U)})\}, \{s_{r_{AP}^{(1)}}(p_{AP}^{(1)}), ..., s_{r_{AP}^{(w)}}(p_{AP}^{(w)}), ..., s_{r_{AP}^{(W)}}(p_{AP}^{(W)})\}\rangle$

can be obtained.

Since $\sum_{u=1}^{U} p_{AP}^{(u)} \leqslant 1; \sum_{w=1}^{W} p_{AP}^{(w)} \leqslant 1$, the normalized DPLTS is now needed for

$\langle \{s_{r_{AP}^{(1)}}(p_{AP}^{(1)}), ..., s_{r_{AP}^{(u)}}(p_{AP}^{(u)}), ..., s_{r_{AP}^{(U)}}(p_{AP}^{(U)})\}, \; \{s_{r_{AP}^{(1)}}(p_{AP}^{(1)}), ..., s_{r_{AP}^{(w)}}(p_{AP}^{(w)}), ..., s_{r_{AP}^{(W)}}(p_{AP}^{(W)})\}\rangle$

The normalized DPLTS is given below.

$\langle \{s_{r_{AP}^{(1)}}(p_{AP}^{*(1)}), ..., s_{r_{AP}^{(u)}}(p_{AP}^{*(u)}), ..., s_{r_{AP}^{(U)}}(p_{AP}^{*(U)})\},$

$\{s_{r_{AP}^{(1)}}(p_{AP}^{*(1)}), ..., s_{r_{AP}^{(w)}}(p_{AP}^{*(w)}), ..., s_{r_{AP}^{(W)}}(p_{AP}^{*(W)})\}\rangle$

Using this, we can derive DPLTS $D_{AP}(p)$ with probability-adjustment.

STEP 3 If the weighting vector $w = (w_1, w_2, ...w_n)^T$ of criteria is known for $w_i \geqslant 0$ with $\sum_{j=1}^{n} w_j = 1$,, then we use DPLWA operator for the aggregation of DPLTSs against each alternative $B_i (i = 1, 2, ..., m)$.The collective criterion values can be found as follows:

$$Z_i(W) = DPLWA(D_{1i}(p), D_{2i}(p), ..., D_{ni}(p)) \; = w_1 D_{1i}(p) \oplus w_2 D_{2i}(p) \oplus ... \oplus w_n D_{ni}(p) \tag{3.1}$$

STEP 4 Using this method, we can create the score functions for the collective criterion values. For DPLE $D(p) = \langle M(p), N(p)\rangle$ the score function calculated as:

$$E\,(D) = S_{\varphi\text{-}\phi} \tag{3.2}$$

Where,

$$\varphi = \frac{\sum_{k=1}^{\#M(p)} r^{(k)} p^{(k)}}{\sum_{k=1}^{\#M(p)} p^{(k)}}, \qquad \phi = \frac{\sum_{l=1}^{\#N(p)} r^{(l)} p^{(l)}}{\sum_{l=1}^{\#N(p)} p^{(l)}}$$

STEP 5 Obtain the final ranking results of all alternatives.

4. Numerical Example

After downloading an app from the Play Store, users can evaluate its quality using a Double-Hierarchy Probabilistic Linguistic Term Set (DPLTS). For the reference domain, the 50 most popular apps were selected, and evaluations were collected from 40 existing reviewers. Once a user has evaluated some apps, they may also be added as a potential reviewer to evaluate future apps.

Apps outside the reference domain can be evaluated in two ways:

(1) using the DPLTS directly, or

(2) by comparison with the already evaluated apps from the reference domain.

In this study, four science-fiction apps published in 2020 are being evaluated, along with four new potential reviewers. The linguistic term set is defined as $S = \{s_g : g = -3, -2, -1, 0, 1, 2, 3\}$.

For the reference domain apps, the average evaluation value of all potential reviewers is $d = 1.29$. Based on the evaluations provided by four reviewers for the science-fiction apps, their average deviations from the reference domain are $d_1 = 1.5, d_2 = 0.5, d_3 = 1.67, d_4 = 1.5$. The original evaluations of the reference domain apps by reviewers are shown in Table 1. For example, a reference app D_{*1} evaluated by reviewer E_1 is represented as $D_{*1} =< \{s_2(1)\}, \{s_3(1)\} >$. Furthermore, if reviewers feel they are unable to provide a fair and unbiased evaluation for a particular app, they may choose to abstain from rating it. This abstention is denoted by the symbol "–".

Table 1. A reviewer's evaluation based on DPLTS

	B_1	B_2
E_1	$< \{s_{-1}(0.3), s_0(0.7)\}, \{s_{-2}(1)\} >$	$< \{s_2(0.2), s_3(0.8)\}, \{s_{-3}(0.4), s_{-2}(0.6)\} >$
E_2	$< \{s_{-2}(1)\}, \{s_1(0.8), s_2(0.2)\} >$	$< \{s_2(0.5), s_3(0.5)\}, \{s_1(0.9), s_2(0.1)\} >$
E_3	–	$< \{s_3(1)\}, \{s_0(0.2), s_1(0.8)\} >$
E_4	$< \{s_{-1}(0.3), s_0(0.7)\}, \{s_0(0.2), s_1(0.8)\} >$	$< \{s_2(0.2), s_3(0.8)\}, \{s_1(0.4), s_2(0.6)\} >$
	B_3	B_4
E_1	Between D_{*1}& D_{11}	$< \{s_{-2}(1)\}, \{s_1(0.3), s_2(0.7)\} >$
E_2	$< \{s_0(0.8), s_1(0.2)\}, \{s_{-2}(1)\} >$	–
E_3	–	$< \{s_{-1}(0.8), s_0(0.2)\}, \{s_1(0.3), s_2(0.7)\} >$
E_4	$< \{s_1(1)\}, \{s_2(0.3), s_3(0.7)\} >$	Lower than D_{41}

STEP 1. By using the way as explained in step 1, the corrected evaluations are given in Table 2.

Table 2. Corrected evalution

	B_1	B_2
E_1	$< \{s_{-1}(0.51), s_0(0.49)\}, \{s_{-3}(0.21), s_{-2}(0.79)\} >$	$< \{s_2(0.41), s_3(0.59)\}, \{s_{-3}(0.61), s_{-2}(0.39)\} >$
E_2	$< \{s_{-2}(0.21), s_{-1}(0.79)\}, \{s_1(0.01), s_2(0.99)\} >$	$< \{s_3(1)\}, \{s_1(0.11), s_2(0.89)\} >$
E_3	–	$< \{s_2(0.38), s_3(0.62)\}, \{s_0(0.58), s_1(0.42)\} >$
E_4	$< \{s_{-1}(0.51), s_0(0.49)\}, \{s_0(0.41), s_1(0.59)\} >$	$< \{s_2(0.41), s_3(0.59)\}, \{s_1(0.61), s_2(0.39)\} >$
	B_3	B_4
E_1	Between D_{*1}& D_{11}	$< \{s_{-3}(0.21), s_{-2}(0.79)\}, \{s_1(0.99), s_2(0.91)\} >$
E_2	$< \{s_0(0.01), s_1(0.99)\}, \{s_{-2}(0.21), s_{-1}(0.79)\} >$	–
E_3	–	$< \{s_{-2}(0.18), s_{-1}(0.82)\}, \{s_1(0.68), s_2(0.32)\} >$
E_4	$< \{s_0(0.21), s_1(0.79)\}, \{s_2(0.51), s_3(0.49)\} >$	Lower than D_{41}

The computation of the assessment of E_1 against the alternative B_1 is given below for reference.

$SC[\langle M_{11}(p), N_{11}(p)\rangle] = \langle\{s_{r_{11}}(p_{11}^{(1)}), s_{r_{11+1}}(p_{11}^{(2)})\}, \{s_{q_{11}}(p_{11}^{(1)}), s_{q_{11+1}}(p_{11}^{(2)})\}\rangle$

Subject to

$$\left[(-1)\cdot\ 0.3 + 0\cdot 0.7 -\ \left[r_{11}\ \cdot\ p_{11}^{(1)} -\ (r_{11}+1)\ \cdot p_{11}^{(2)}\right]\right] = 1.5 - 1.29 = 0.21,$$

$$p_{11}^{(1)} = 0.3 + 0.21 = 0.51, \quad p_{11}^{(2)} = 0.7 - 0.21 = 0.49,$$

$$p_{11}^{(1)} + p_{11}^{(2)} = 1$$

and

$$\left[(-3)\cdot\ 0 + (-2)\cdot\ 1 - \left[q_{11}\ \cdot\ p_{11}^{(1)} -\ (q_{11}+1)\ \cdot\ p_{11}^{(2)}\right]\right] = 1.5 - 1.29 = 0.21,$$

$$p_{11}^{(1)} = 0 + 0.21 = 0.21, \quad p_{11}^{(2)} = 1 - 0.21 = 0.79,$$

$$p_{ij}^{(1)} + p_{ij}^{(2)} = 1$$

So, $\langle M_{11}(p), N_{11}(p)\rangle = D_{11} = < \{s_{-1}(0.51), s_0(0.49)\}, \{s_{-3}(0.21), s_{-2}(0.79)\} >$

Similarly, the corrected evaluations are presented in Table 2 , which contains all of the corrected evaluations.

Step 2. Convert CLEs into DPLTSs.

E_{g_h} (Between D_{*1}&D_{11})
$D_{*1} =< \{s_2(1)\}, \{s_3(1)\} >$,
$D_{11} =< \{s_{-1}(0.51), s_0(0.49)\}, \{s_{-3}(0.21), s_{-2}(0.79)\} >$
Firstly , we find $E_{\acute{g}_h}$(Between M_{*1} & M_{11})
Where $M_{*1}^{(1)} = \{s_2(1)\}$ and $M_{11}^{(1)} = \{s_{-1}(0.51), s_0(0.49)\}$
$= \{ \sum_{i=1}^{\#M_1(p)} \sum_{j=1}^{\#M_2(p)} p_1^{(i)} p_2^{(j)} . E_{\acute{g}_h}(\text{between } M_{*1}^{(1)} \text{ and } M_{11}^{(1)})\}$
$=1 \cdot 0.51\{s_{-1}(\frac{1}{3}), s_0(\frac{1}{3}), s_1(\frac{1}{3})\} \oplus 1 \cdot 0.49\{s_0(\frac{1}{2}), s_1(\frac{1}{2})\}$
$= \{s_{-1}(\frac{0.51}{3}),\ s_0(\frac{0.51}{3} + \frac{0.49}{2}), s_1(\frac{0.51}{3} + \frac{0.49}{2})\}$
$= \{s_{-1}(0.170), s_0(0.415), s_1(0.415)\}$
Now
$E_{\acute{g}_h}$(Between N_{*1} & N_{11})
Where $N_{*1}^{(2)} = \{s_3(1)\}$ and $N_{11}^{(2)} = \{s_{-3}(0.21), s_{-2}(0.79)\}$
$\{ \sum_{u=1}^{\#N_1(p)} \sum_{v=1}^{\#N_2(p)} p_1^{(u)} p_2^{(v)} \ . \ E_{\acute{g}_h}(\text{between } N_{*1}^{(2)} \text{ and } N_{11}^{(2)})\}$
$=1 \cdot 0.21\{s_{-3}(\frac{1}{6}), s_{-2}(\frac{1}{6}), s_{-1}(\frac{1}{6}), s_0(\frac{1}{6}), s_1(\frac{1}{6}), s_2(\frac{1}{6})\}$
$\oplus\ 1 \cdot 0.79\{s_{-2}(\frac{1}{5}), s_{-1}(\frac{1}{5}), s_0(\frac{1}{5}), s_1(\frac{1}{5}), s_2(\frac{1}{5})\}$
$= \{s_{-3}(\frac{0.21}{6}), s_{-2}(\frac{0.21}{6} + \frac{0.79}{5}), s_{-1}(\frac{0.21}{6} + \frac{0.79}{5}),$
$s_0(\frac{0.21}{6} + \frac{0.79}{5}), s_1(\frac{0.21}{6} + \frac{0.79}{5}), s_2(\frac{0.21}{6} + \frac{0.79}{5})\}$
$= \{s_{-3}(0.035), s_{-2}(0.193), s_{-1}(0.193), s_0(0.193), s_1(0.193), s_2(0.193)\}$
E_{g_h}(Between D_{*1} & D_{11}) $= \langle \{s_{-1}(0.170), s_0(0.415), s_1(0.415)\},$
$\{s_{-3}(0.035), s_{-2}(0.193), s_{-1}(0.193), s_0(0.193), s_1(0.193), s_2(0.193)\}\rangle$
To calculate the PA operators, follow these steps:
Function η sets $k, b = 1$ in this example.
For $\{s_{-1}(0.170), s_0(0.415), s_1(0.415)\}$ and $\{s_0(0.01), s_1(0.99)\}$
$\eta(-1, 0) = -2,\ \ \eta(0, 0) = 0, \eta(1, 0) = -2,$
$\eta(-1, 1) = -3,\ \ \ \eta(0, 1) = -1,\ \ \ \eta(1, 1) = -1,$
By using $PA(s_{r_E^{(u)}}, s_{r_i^{(v)}}) = \frac{\eta\left(r_E^{(u)}\right)}{\sum_{u=1}^{U} \eta\left(r_E^{(u)}\right)}$
$PA(-1, 0) = \frac{1}{2},\ \ PA(0, 0) = 0,\ \ PA(1, 0) = \frac{1}{2},$
$PA(-1, 1) = \frac{3}{5},\ \ PA(0, 1) = \frac{1}{5},\ PA(1, 1) = \frac{1}{5},$
Similarly, for
$\{s_{-3}(0.035), s_{-2}(0.193), s_{-1}(0.193), s_0(0.193), s_1(0.193), s_2(0.193)\}$
and $\{s_{-2}(0.21), s_{-1}(0.79)\}$
$\eta(-3, -2) = -5, \eta(-2, -2) = -3, \eta(-1, -2) = -3,$
$\eta(0, -2) = -4, \eta(1, -2) = -6, \eta(2, -2) = -8,$
$\eta(-3, -1) = -6, \eta(-2, -1) = -4, \eta(-1, -1) = -2,$
$\eta(0, -1) = -3, \eta(1, -1) = -5, \eta(2, -1) = -7,$
By using $PA(s_{r_E^{(w)}}, s_{r_i^{(y)}}) = \frac{\eta\left(r_E^{(w)}\right)}{\sum_{w=1}^{W} \eta\left(r_E^{(w)}\right)}$
$PA(-3, -2) = \frac{5}{29}, PA(-2, -2) = \frac{3}{29}, PA(-1, -2) = \frac{3}{29},$
$PA(0, -2) = \frac{4}{29}, PA(1, -2) = \frac{6}{29}, PA(2, -2) = \frac{8}{29},$
$PA(-3, -1) = \frac{2}{9}, PA(-2, -1) = \frac{4}{27}, PA(-1, -1) = \frac{2}{27},$
$PA(0, -1) = \frac{1}{9}, PA(1, -1) = \frac{5}{27}, PA(2, -1) = \frac{7}{27},$
Now calculate adjusted probability
For$\{s_{-1}(0.170), s_0(0.415), s_1(0.415)\},$
$\{s_0(0.01), s_1(0.99)\}, \{s_{-3}(0.21), s_{-2}(0.79)\}$
and
$PA(-1, 0) = \frac{1}{2},\ \ PA(0, 0) = 0,\ \ PA(1, 0) = \frac{1}{2},$

$PA(-1,1) = \frac{3}{5}, \ PA(0,1) = \frac{1}{5}, \ PA(1,1) = \frac{1}{5},$
By using $p_{AP}^{(u)} = \frac{1}{N}\sum_{i=1}^{N}\sum_{\nu=1}^{V} p_i^{(\nu)}.PA(s_{r_E^{(u)}}, s_{r_i^{(\nu)}}).p_E^{(u)}$
$p_{AP}^{(1)} = \frac{1}{2}[(0.01 \cdot \frac{1}{2} \cdot 0.170 + 0.99 \cdot \frac{3}{5} \cdot 0.170) + (0.21 \cdot \frac{1}{2} \cdot 0.170 + 0.79 \cdot \frac{3}{5} \cdot 0.170)] = 0.1001$
$p_{AP}^{(2)} = \frac{1}{2}[(0.01 \cdot 0 \cdot 0.415 + 0.99 \cdot \frac{1}{5} \cdot 0.415) + (0.21 \cdot 0 \cdot 0.415 + 0.79 \cdot \frac{1}{5} \cdot 0.415)] = 0.2382$
$p_{AP}^{(3)} = \frac{1}{2}[(0.01 \cdot \frac{1}{2} \cdot 0.415 + 0.99 \cdot \frac{1}{5} \cdot 0.415) + (0.21 \cdot \frac{1}{2} \cdot 0.415 + 0.79 \cdot \frac{1}{5} \cdot 0.415)] = 0.0966$
$\Longrightarrow \{s_{-1}(0.1001), s_0(0.2382), s_1(0.0966)\}$
Similarly for,
$\{s_{-3}(0.035), s_{-2}(0.193), s_{-1}(0.193), s_0(0.193), s_1(0.193), s_2(0.193)\},$
$\{s_{-2}(0.21), s_{-1}(0.79)\}, \{s_1(0.09), s_2(0.91)\}$
and
$PA(-3,-2) = \frac{5}{29}, PA(-2,-2) = \frac{3}{29}, PA(-1,-2) = \frac{3}{29},$
$PA(0,-2) = \frac{4}{29}, PA(1,-2) = \frac{6}{29}, PA(2,-2) = \frac{8}{29},$
$PA(-3,-1) = \frac{2}{9}, PA(-2,-1) = \frac{4}{27}, PA(-1,-1) = \frac{2}{27},$
$PA(0,-1) = \frac{1}{9}, PA(1,-1) = \frac{5}{27}, PA(2,-1) = \frac{7}{27},$
By using $p_{AP}^{(w)} = \frac{1}{N}\sum_{i=1}^{N}\sum_{y=1}^{Y} p_i^{(y)}.PA(s_{r_E^{(w)}}, s_{r_i^{(y)}}).p_E^{(w)}$
$p_{AP}^{(1)} = \frac{1}{2}[(0.21 \cdot \frac{5}{29} \cdot 0.035 + 0.79 \cdot \frac{2}{9} \cdot 0.035) + (0.09 \cdot \frac{5}{29} \cdot 0.035 + 0.91 \cdot \frac{2}{9} \cdot 0.035)] = 0.0075$
$p_{AP}^{(2)} = \frac{1}{2}[(0.21 \cdot \frac{3}{29} \cdot 0.193 + 0.79 \cdot \frac{4}{27} \cdot 0.193) + (0.09 \cdot \frac{3}{29} \cdot 0.193 + 0.91 \cdot \frac{4}{27} \cdot 0.193) = 0.0272$
$p_{AP}^{(3)} = \frac{1}{2}[(0.21 \cdot \frac{3}{29} \cdot 0.193 + 0.79 \cdot \frac{2}{27} \cdot 0.193) + (0.09 \cdot \frac{3}{29} \cdot 0.193 + 0.91 \cdot \frac{2}{27} \cdot 0.193) = 0.0151$
$p_{AP}^{(4)} = \frac{1}{2}[(0.21 \cdot \frac{4}{29} \cdot 0.193 + 0.79 \cdot \frac{1}{9} \cdot 0.193) + (0.09 \cdot \frac{4}{29} \cdot 0.193 + 0.91 \cdot \frac{1}{9} \cdot 0.193) = 0.0222$
$p_{AP}^{(5)} = \frac{1}{2}[(0.21 \cdot \frac{6}{29} \cdot 0.193 + 0.79 \cdot \frac{5}{27} \cdot 0.193) + (0.09 \cdot \frac{6}{29} \cdot 0.193 + 0.91 \cdot \frac{5}{27} \cdot 0.193) = 0.0363$
$p_{AP}^{(6)} = \frac{1}{2}[(0.21 \cdot \frac{8}{29} \cdot 0.193 + 0.79 \cdot \frac{7}{27} \cdot 0.193) + (0.09 \cdot \frac{8}{29} \cdot 0.193 + 0.91 \cdot \frac{7}{27} \cdot 0.193) = 0.0505$
$\Longrightarrow \{s_{-3}(0.0075), s_{-2}(0.0272), s_{-1}(0.0151), s_0(0.0222), s_1(0.0363), s_2(0.0505)\}$
The adjusted DPLTS is obtained
$= \langle\{s_{-1}(0.1001), s_0(0.2382), s_1(0.0966)\},$
$\{s_{-3}(0.0075), s_{-2}(0.0272), s_{-1}(0.0151), s_0(0.0222), s_1(0.0363), s_2(0.0505)\}\rangle$
Normalization
Since, $\sum_{u=1}^{U} p_{AP}^{(u)} \leqslant 1; \ \sum_{w=1}^{W} p_{AP}^{(w)} \leqslant 1$, the normalization is needed:
$\langle\{s_{-1}(0.1001), s_0(0.2382), s_1(0.0966)\},$
$\{s_{-3}(0.0075), s_{-2}(0.0272), s_{-1}(0.0151), s_0(0.0222), s_1(0.0363), s_2(0.0505)\}\rangle$
The normalized DPLTS is given below with separate componensts.
$\langle\{s_{-1}(0.230), s_0(0.548), s_1(0.222)\},$
$\{s_{-3}(0.047), s_{-2}(0.171), s_{-1}(0.095), s_0(0.140), s_1(0.229), s_2(0.318)\}\rangle$
E_{g_h} (Lower than D_{41})
$D_{41} =< \{s_{-1}(0.51), s_0(0.49)\}, \{s_0(0.41), s_1(0.59)\} >$
Where, $M_{41} = \{s_{-1}(0.51), s_0(0.49)\}, \ \ N_{41} = \{s_0(0.41), s_1(0.59)\}$
For $E_{g_h}(\text{Lower than } M_{41}) = \{ \sum_{i=1}^{\#M_1(p)} p_1^{(i)}.E_{\acute{g}_h}(\text{lower than } M_{41}^{(i)})\}$
$= 0.51 \cdot \{s_{-3}(\frac{1}{3}), s_{-2}(\frac{1}{3}), s_{-1}(\frac{1}{3})\} \oplus 0.49 \cdot \{s_{-3}(\frac{1}{4}), s_{-2}(\frac{1}{4}), s_{-1}(\frac{1}{4}), s_0(\frac{1}{4})\}$
$= \{s_{-3}(\frac{0.51}{3} + \frac{0.49}{4}), s_{-2}(\frac{0.51}{3} + \frac{0.49}{4}), s_{-1}(\frac{0.51}{3} + \frac{0.49}{4}), s_0(\frac{0.49}{4})\}$
$= \{s_{-3}(0.2925), s_{-2}(0.2925), s_{-1}(0.2925), s_0(0.1225)\}$

Similarly,
$E_{g_h}(\text{Lowerthan } N_{41}) = \{ \sum_{u=1}^{\#N_1(p)} p_1^{(u)}.E_{\acute{g}_h}(\text{lower than } N_1^{(u)})\}$
$= 0.41 \cdot \{s_{-3}(\frac{1}{4}), s_{-2}(\frac{1}{4}), s_{-1}(\frac{1}{4}), s_0(\frac{1}{4})\} \oplus 0.59 \cdot \{s_{-3}(\frac{1}{5}), s_{-2}(\frac{1}{5}), s_{-1}(\frac{1}{5}), s_0(\frac{1}{5}), s_1(\frac{1}{5})\}$
$= \{s_{-3}(\frac{0.41}{4} + \frac{0.59}{5}), s_{-2}(\frac{0.41}{4} + \frac{0.59}{5}), s_{-1}(\frac{0.41}{4} + \frac{0.59}{5}), s_0(\frac{0.41}{4} + \frac{0.59}{5}), s_1(\frac{0.59}{5})\}$
$= \{s_{-3}(0.2205), s_{-2}(0.2205), s_{-1}(0.2205), s_0(0.2205), s_1(0.118)\}$
$\Longrightarrow E_{g_h}(\text{Lower than } D_{41}) = \langle\{s_{-3}(0.2925), s_{-2}(0.2925), s_{-1}(0.2925), s_0(0.1225)\},$

$\{s_{-3}(0.2205), s_{-2}(0.2205), s_{-1}(0.2205), s_0(0.2205), s_1(0.118)\}\rangle$

To calculate the PA operators, follow these steps:

Function η sets $k, b = 1$ in this example.

For, $\{s_{-3}(0.2925), s_{-2}(0.2925), s_{-1}(0.2925), s_0(0.1225)\} and \{s_{-2}(0.18), s_{-1}(0.82)\}$

$\eta(-3,-2) = -3, \eta(-2,-2) = -1, \eta(-1,-2) = -2, \eta(0,-2) = -4$

$\eta(-3,-1) = -4, \eta(-2,-1) = -2, \eta(-1,-1) = -1, \eta(0,-1) = -3$

by using

$PA(s_{r_E^{(u)}}, s_{r_i^{(v)}}) = \frac{\eta\left(r_E^{(u)}\right)}{\sum_{u=1}^{U} \eta\left(r_E^{(u)}\right)}$

$PA(-3,-2) = \frac{3}{10}, PA(-2,-2) = \frac{1}{10}, PA(-1,-2) = \frac{1}{5}, PA(0,-2) = \frac{2}{5},$

$PA(-3,-1) = \frac{2}{5}, PA(-2,-1) = \frac{1}{5}, PA(-1,-1) = \frac{1}{10}, PA(0,-1) = \frac{3}{10},$

Similarly for,

$\{s_{-3}(0.2205), s_{-2}(0.2205), s_{-1}(0.2205), s_0(0.2205), s_1(0.118)\} and \{s_1(0.68), s_2(0.32)\}$

$\eta(-3,1) = -7, \eta(-2,1) = -5, \eta(-1,1) = -3, \eta(0,1) = -3, \eta(1,1) = -3,$

$\eta(-3,2) = -8, \eta(-2,2) = -6, \eta(-1,2) = -4, \eta(0,2) = -4, \eta(1,2) = -4,$

by using

$PA(s_{r_E^{(w)}}, s_{r_i^{(y)}}) = \frac{\eta\left(r_E^{(w)}\right)}{\sum_{w=1}^{W} \eta\left(r_E^{(w)}\right)}$

$PA(-3,1) = \frac{1}{3}, PA(-2,1) = \frac{5}{27}, PA(-1,1) = \frac{1}{7}, PA(0,1) = \frac{1}{7}, PA(1,1) = \frac{1}{7},$

$PA(-3,2) = \frac{4}{13}, PA(-2,2) = \frac{3}{13}, PA(-1,2) = \frac{2}{13}, PA(0,2) = \frac{2}{13}, PA(1,2) = \frac{2}{13},$

Now calculate adjusted probability

For $\{s_{-3}(0.2925), s_{-2}(0.2925),$
$s_{-1}(0.2925), s_0(0.1225)\}, \{s_{-2}(0.18), s_{-1}(0.82)\}$

and

$PA(-3,-2) = \frac{3}{10}, PA(-2,-2) = \frac{1}{10}, PA(-1,-2) = \frac{1}{5}, PA(0,-2) = \frac{2}{5},$

$PA(-3,-1) = \frac{2}{5}, PA(-2,-1) = \frac{1}{5}, PA(-1,-1) = \frac{1}{10}, PA(0,-1) = \frac{3}{10},$

$p_{AP}^{(1)} = (0.18 \cdot \frac{3}{10} \cdot 0.2925 + 0.82 \cdot \frac{2}{9} \cdot 0.2925) = 0.112$

$p_{AP}^{(2)} = (0.18 \cdot \frac{1}{10} \cdot 0.2925 + 0.82 \cdot \frac{1}{5} \cdot 0.2925) = 0.053$

$p_{AP}^{(3)} = (0.18 \cdot \frac{1}{5} \cdot 0.2925 + 0.82 \cdot \frac{1}{10} \cdot 0.2925) = 0.035$

$p_{AP}^{(1)} = (0.18 \cdot \frac{2}{5} \cdot 0.1225 + 0.82 \cdot \frac{3}{10} \cdot 0.1225) = 0.039$

$\Longrightarrow \{s_{-3}(0.112), s_{-2}(0.053), s_{-1}(0.035), s_0(0.039)\}$

Similarly for,

$\{s_{-3}(0.2205), s_{-2}(0.2205), s_{-1}(0.2205), s_0(0.2205), s_1(0.118)\},$
$\{s_1(0.68), s_2(0.32)\}$

and

$PA(-3,1) = \frac{1}{3}, PA(-2,1) = \frac{5}{27}, PA(-1,1) = \frac{1}{7}, PA(0,1) = \frac{1}{7}, PA(1,1) = \frac{1}{7},$

$PA(-3,2) = \frac{4}{13}, PA(-2,2) = \frac{3}{13}, PA(-1,2) = \frac{2}{13}, PA(0,2) = \frac{2}{13}, PA(1,2) = \frac{2}{13},$

$p_{AP}^{(1)} = (0.68 \cdot \frac{1}{3} \cdot 0.2205 + 0.32 \cdot \frac{4}{13} \cdot 0.2205) = 0.072$

$p_{AP}^{(2)} = (0.68 \cdot \frac{5}{27} \cdot 0.2205 + 0.32 \cdot \frac{3}{13} \cdot 0.2205) = 0.044$

$p_{AP}^{(3)} = (0.68 \cdot \frac{1}{7} \cdot 0.2205 + 0.32 \cdot \frac{2}{13} \cdot 0.2205) = 0.032$

$p_{AP}^{(4)} = (0.68 \cdot \frac{1}{7} \cdot 0.2205 + 0.32 \cdot \frac{2}{13} \cdot 0.2205) = 0.032$

$p_{AP}^{(5)} = (0.68 \cdot \frac{1}{7} \cdot 0.118 + 0.32 \cdot \frac{2}{13} \cdot 0.118) = 0.017$

$\Longrightarrow \{s_{-3}(0.072), s_{-2}(0.044), s_{-1}(0.032), s_0(0.032), s_1(0.017)\}$

The adjusted DPLTS is obtained as follows:

$\langle\{s_{-3}(0.112), s_{-2}(0.053), s_{-1}(0.035), s_0(0.039)\},$
$\{s_{-3}(0.072), s_{-2}(0.044), s_{-1}(0.032), s_0(0.032), s_1(0.017)\}\rangle$

Normalization

Since, $\sum_{u=1}^{U} p_{AP}^{(u)} \leqslant 1; \sum_{w=1}^{W} p_{AP}^{(w)} \leqslant 1$, the normalization is needed:
$\langle\{s_{-3}(0.112), s_{-2}(0.053), s_{-1}(0.035), s_{0}(0.039)\},$
$\{s_{-3}(0.072), s_{-2}(0.044), s_{-1}(0.032), s_{0}(0.032), s_{1}(0.017)\}$
The normalized DPLTS is given below with separate componensts.
$\langle\{s_{-3}(0.47), s_{-2}(0.22), s_{-1}(0.15), s_{0}(0.16)\},$
$\{s_{-3}(0.37), s_{-2}(0.22), s_{-1}(0.16), s_{0}(0.16), s_{1}(0.09)\}\rangle$

Table 3. Transformed evaluations

	B_1	B_2
E_1	$<\{s_{-1}(0.51), s_0(0.49)\},\{s_{-3}(0.21), s_{-2}(0.79)\}>$	$<\{s_2(0.41), s_3(0.59)\},\{s_{-3}(0.61), s_{-2}(0.39)\}>$
E_2	$<\{s_{-2}(0.21), s_{-1}(0.79)\},\{s_1(0.01), s_2(0.99)\}>$	$<\{s_3(1)\},\{s_1(0.11), s_2(0.89)\}>$
E_3	−	$<\{s_2(0.38), s_3(0.62)\},\{s_0(0.58), s_1(0.42)\}>$
E_4	$<\{s_{-1}(0.51), s_0(0.49)\},\{s_0(0.41), s_1(0.59)\}>$	$<\{s_2(0.41), s_3(0.59)\},\{s_1(0.61), s_2(0.39)\}>$
	B_3	B_4
E_1	$<\{s_{-1}(0.23), s_0(0.55), s_1(0.22)\},\{s_{-3}(0.05), s_{-2}(0.17), s_{-1}(0.10), s_0(0.14), s_1(0.22), s_2(0.32)\}>$	$<\{s_{-3}(0.21), s_{-2}(0.79)\},\{s_1(0.99), s_2(0.01)\}>$ −
E_2	$<\{s_0(0.01), s_1(0.99)\},\{s_{-2}(0.21), s_{-1}(0.79)\}>$	$<\{s_{-2}(0.18), s_{-1}(0.82)\},$
E_3	−	$\{s_1(0.68), s_2(0.32)\}>$
E_4	$<\{s_0(0.21), s_1(0.79)\},\{s_2(0.51), s_3(0.49)\}>$	$<\{s_{-3}(0.47), s_{-2}(0.22), s_{-1}(0.15), s_0(0.16)\},$ $<\{s_{-3}(0.47), s_{-2}(0.22), s_{-1}(0.15), s_0(0.16)\}$

Step 3. Calculate the collective criterion values $Z_t(W)(t = 1, 2, 3, 4)$
$Z_i(W) = DPLWA(D_{1i}(p), D_{2i}(p), ..., D_{ni}(p))$

$= w_1 D_{1i}(p) \oplus w_2 D_{2i}(p) \oplus ... \oplus w_n D_{ni}(p)$

$Z_1(W) = \langle\{s_{-0.3754}(0.0546), s_{0.4419}(0.0525), s_{-0.1569}(0.2055), s_{0.6075}(0.1974),$
$s_{-0.1867}(0.0525), s_{0.5849}(0.0504), s_{0.0196}(0.1974), s_{0.7413}(0.1897)\}, \{s_{-0.2704}$
$(0.0009), s_{0.2192}(0.0012), s_{0.3437}(0.0852), s_{0.7413}(00.1227), s_{-0.1533}(0.0032),$
$s_{0.3188}(0.0047), s_{0.4386}(0.3207), s_{0.8221}(0.4614)\}\rangle$

$Z_2(W) = \langle\{s_3(0.0639), s_3(0.0919), s_3(0.1042), s_3(0.1500), s_3(0.0919), s_3(0.1323),$
$s_3(0.1500), s_3(0.2158)\}, \{s_{0.4055}(0.0237), s_{1.0337}(0.0152), s_{0.5085}(0.0172), s_{1.1118}$
$(0.0110), s_{0.8926}(0.1921), s_{1.4028}(0.1228), s_{0.9763}(0.1391), s_{1.4663}(0.0889), s_{0.4983}$
$(0.0152), s_{1.1041}(0.0097), s_{0.5977}(0.0110), s_{1.1794}(0.0070), s_{0.9680}(0.1228), s_{1.4600}$
$(0.0785), s_{1.0488}(0.0889), s_{1.5212}(0.0569)\}\rangle$
$Z_3(W) = \langle\{s_{0.2258}(0.0054), s_3(0.0051), s_{0.4054}(0.0201), s_3(0.0194), s_{0.3251}(0.0182),$
$s_3(0.0175), s_{0.4983}(0.0685), s_3(0.0568), s_{0.4419}(0.0107), s_3(0.0103), s_{0.6075}(0.0403),$
$s_3(0.0387), s_{0.5849}(0.0150), s_3(0.0144), s_{0.7413}(0.0564), s_3(0.0542), s_{0.7730}(0.0236),$
$s_3(0.0226), s_{0.9172}(0.0886), s_3(0.0582), s_{1.0613}(0.0343), s_3(0.0329), s_{1.1868}(0.1289),$
$s_3(0.1239)\}, \{s_{0.4419}(0.0054), s_3(0.0051), s_{0.6075}(0.0201), s_3(0.0194), s_{0.5849}(0.0182),$
$s_3(0.0175), s_{0.7413}(0.0685), s_3(0.0568), s_{0.7730}(0.0107), s_3(0.0103), s_{0.9172}(0.040),$
$s_3(0.0387), s_{0.5627}(0.0150), s_3(0.0144), s_{0.7205}(0.0564), s_3(0.0542), s_{0.4571}(0.0236),$
$s_3(0.0226), s_{0.6218}(0.0886), s_3(0.0582), s_{0.4777}(0.0343), s_3(0.0329), s_{0.6410}(0.1289),$
$s_3(0.1239)\}\rangle$
$Z_4(W) = \langle\{s_{-2.8916}(0.0178), s_{-2.4772}(0.0083), s_{-2.0095}(0.0057), s_{-1.4649}(0.0060), s_{-2.7616}$
$(0.0809), s_{-2.3564}(0.0379), s_{-1.8990}(0.0258), s_{-1.3664}(0.0276), s_{-2.6807}(0.0668), s_{-2.2812}$
$(0.0313), s_{-1.8302}(0.0213), s_{-1.3051}(0.0228), s_{-2.5553}(0.3045), s_{-2.1646}(0.1425), s_{-1.7236}$
$(0.0972), s_{-1.2101}(0.1036)\}, \{s_{1.2081}(0.0226), s_{1.3341}(0.0135), s_{1.4764}(0.0098), s_{1.6420}$
$(0.0098), s_{1.8453}(0.0055), s_{1.3281}(0.0107), s_{1.4457}(0.0063), s_{1.5784}(0.0046), s_{1.7330}(0.0046),$
$s_{1.9226}(0.0026), s_{2.1040}(0.2290), s_{2.1670}(0.1361), s_{2.2381}(0.0990), s_{2.3210}(0.0990), s_{2.4226}$
$(0.0557), s_{2.1640}(0.1077), s_{2.2228}(0.0641), s_{2.2892}(0.0466), s_{2.3664}(0.0466), s_{2.4613}(0.0262)\}\rangle$

STEP 4 Calculate the score functions of the collective criterion values $Sc(Z_t(w))(t = 1, 2, 3, 4)$

$$Sc(Z_1(w) = s_{-0.3867}, Sc(Z_2(w) = s_{1.8974}, Sc(Z_3(w) = s_{0.0860}, Sc(Z_4(w)) = s_{-1.9804}$$

STEP 5. Get the final ranking for each alternative $B_s(s = 1, 2, 3, 4) : B_2 > B_3 >, B_1 > B_4$. As a result, B_2 should be the best option.

4.1. *Results and discussion*

The proposed decision-making process is applied to resolve the issue of incomplete assessments and DPLTSs with CLEs. As seen in this example, people tend to evaluate an app after they have used it several days, and the process usually takes some time. It may take weeks, months, or even years to evaluate two apps at the same time. It is possible that the reviewer does not remember the specific evaluation of the app that was previously given in the form of DPLTS. However, it is extremely impractical to expect reviewers to use all apps and provide feedback. Hence, when supporting linguistic expressions of comparison and incomplete assessments, the use of DPLTS for evaluation is practical. Evaluation and decision making can be made more accurate and flexible by loosening the requirements for evaluators. Nevertheless, DPLTSs provide a more realistic picture of the decision makers and allow them to make predictions more easily. Consequently, when it comes to comparing new apps with respect to their advantages, our method yields more reliable and reasonable results. As a first step, we solve above example with proposed method using the dual probabilistic linguistic weighted average operator with CLEs and incomplete assessments. The outcomes are calculated as follows: $Sc(Z_1(w) = s_{-0.3867}$, $Sc(Z_2(w) = s_{1.8974}$, $Sc(Z_3(w) = s_{0.0860}$, $Sc(Z_4(w)) = s_{-1.9804}$. Moreover, the ranking order is derived as $B_2 > B_3 > B_1 > B_4$ and as a result, B_2 will be the best option.

5. Conclusion

The PLTSs are extended to the DPLTSs, allowing the DMs to express their decision information more conformed to reality through using different perspectives and scenarios. First, When the evaluator fails to provide an accurate assessment, but can suggest comparisons between alternatives, a method is proposed for transforming CLEs into DPLTSs. Here, we outline the principles that govern how the DPLTSs operate. Following that, we propose comparing DPLTSs by means of a score function and an accuracy function. This paper presents a numerical example and corresponding comparative analysis that demonstrate the superiority and validity of the developed methodology. We presented the proposed method under the assumption that the weights of the attributes are known, and there is no discussion of the situations where the attribute weight is unknown or partially known when the proposed method is given. One of the key advantage of our study is that the proposed method for transforming CLEs into DPLTSs with aggregation operator of DPLTSs and provides more convincing arguments in light of both membership and non-membership degrees. As a whole, based on the findings of this paper, more research related to the DPLTSs can be conducted in a future perspective.

References

[1] Alonso, S. Herrera-Viedma, E.; Chiclana, F.; Herrera, F.; (2007). Group Decision-Making Model With Incomplete Fuzzy Preference Relations Based on Additive Consistency. , 37(1), 0–189. doi:10.1109/tsmcb.2006.875872 1

[2] Bai, C., Zhang, R., Qian, L., & Wu, Y. (2017). Comparisons of probabilistic linguistic term sets for multi-criteria decision making. Knowledge-Based Systems, 119,284–291. doi:10.1016/j.knosys.2016.12.020 1

[3] Chen, Y.-H., Wang, T.-C., & Wu, C.-Y. (2011). Multi-criteria decision making with fuzzy linguistic preference relations. Applied Mathematical Modelling, 35(3), 1322–1330. doi:10.1016/j.apm.2010.09.009 1

[4] Durand, M., & Truck, I. (2018). A new proposal to deal with hesitant linguistic expressions on preference assessments. Information Fusion, 41, 176–181. doi: 10.1016/j.inffus.2017.09.007 1

[5] Feng, X., Liu, Q., & Wei, C. (2018). Probabilistic linguistic QUALIFLEX approachwith possibility degree comparison. Journal of Intelligent & Fuzzy Systems, 1–12. doi:10.3233/jifs-172112. 1

[6] Gao, J., Liang, Z., Liao, H., & Xu, Z. (2019). Expected consistency-based emergency decision making with incomplete probabilistic linguistic preference relations, Knowledge-Based Systems. 176, 15-28. doi:10.1016/j.knosys.2019.03.020. 1

[7] Herrera, F. (1995). A sequential selection process in group decision making with a linguistic assessment approach. Information Sciences, 85(4), 223–239. doi:10.1016/0020-0255(95)00025-k 1

[8] Jin, C., Wang, H., & Xu, Z. S. (2019). Uncertain probabilistic linguistic term sets in group decision making. International Journal of Fuzzy Systems, 21(4), 1241– 1258. Retrived from https://link.springer.com/article/10.1007/s40815-019-00619-9 1

[9] Krishankumar, R., Ravichandran, K., Ahmed, M., Kar, S., & Tyagi, S. (2018). Probabilistic Linguistic Preference Relation-Based Decision Framework for Multi-Attribute Group Decision Making. Symmetry, 11(1), 2. doi:10.3390/sym11010002 1

[10] Labella, Á., Rodríguez, R. M., Alzahrani, A. A., & Martínez, L. (2020). A Consensus Model for Extended Comparative Linguistic Expressions with Symbolic Translation. Mathematics, 8(12), 2198. doi:10.3390/math8122198 1

[11] Liao, H., Mi, X., & Xu, Z. (2019). A survey of decision-making methods with probabilistic linguistic information: bibliometrics, preliminaries,methodologies, applications and future directions. Fuzzy Optimization and Decision Making. doi:10.1007/s10700-019-09309-5 1

[12] Liu, Y., Rodriguez, R. M., Hagras, H., Liu, H., Qin, K., & Martinez, L. (2019). Type-2 fuzzy envelope of hesitant fuzzy linguistic term set: a new representation model of comparative linguistic expression. IEEE Transactions on Fuzzy Systems, 1–1. doi:10.1109/tfuzz.2019.2898155 1

[13] Pang, Q., Wang, H., & Xu, Z. (2016). Probabilistic linguistic term sets in multi-attribute group decision making. Information Sciences, 369, 128–143. doi:10.1016/j.ins.2016.06.021 1

[14] Pei, Zheng; Ruan, Da; Liu, Jun; Xu, Yang (2010). [Atlantis Computational Intelligence Systems] Linguistic Values Based Intelligent Information Processing: Theory, Methods, and Applications Volume 1 $\|$., $10.2991/978-94-91216-28-2()$, $--$ .doi : $10.2991/978-94-91216-28-2$1

[15] Q. Pang, H. Wang and Z.S. Xu, Probabilistic linguistic term sets in multi-attribute group decision making, Information Sciences 369 (2016), 128–143. 1, 2.5, 2.6, 2.7

[16] Rodríguez, R. M., Martínez, L., & Herrera, F. (2013). A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. Information Sciences, 241, 28–42. doi:10.1016/j.ins.2013.04.006 1, 3

[17] Song, Y. (2018). Deriving the priority weights from probabilistic linguistic preference relation with unknown probabilities. PLOS ONE, 13(12), e0208855. doi:10.1371/journal.pone.0208855 1

[18] Torra, V. (2010). Hesitant fuzzy sets, International Journals of Intelligent System, 25(6), 529-539. https://doi.org/10.1002/int.20418 2.3

[19] Wu, H., Wang, J., Feng, J., Liu, D., & Liu, S. (2020). Research on the decision-making of flood prevention emergency plans during reservoir construction based on generalized intuitionistic fuzzy soft sets and TOPSIS. Water Supply. doi:10.2166/ws.2020.182

[20] Wu, X., & Liao, H. (2018). A consensus-based probabilistic linguistic gained and lost dominance score method. European Journal of Operational Research. doi: 10.1016/j.ejor.2018.07.044 1

[21] Wang, X., Wang, J., & Zhang, H. (2018). Distance-based multicriteria group decisionmaking approach with probabilistic linguistic term sets. Expert Systems36, e12352. doi:10.1111/exsy.12352 1

[22] Wang, H., Xu, Z., & Zeng, X.-J. (2018). Hesitant fuzzy linguistic term sets for linguistic decision making: Current developments, issues and challenges. Information Fusion, 43, 1–12. doi:10.1016/j.inffus.2017.11.010 2, 2.4

[23] Xie, W. Y., Xu, Z. S., & Ren, Z. L. (2017). Dual probabilistic linguistic term set and its application on multi-criteria group decision making problems. 2017 IEEE International Conference on Industrial Engineering and Engineering Management (IEEM). doi:10.1109/ieem.2017.8290137 1, 2.10, 2.11, 2.12, 2.13, 2.14

[24] Xu, Z., He, Y., & Wang, X. (2018). An overview of probabilistic-based expressions for qualitative decision-making: techniques, comparisons and developments. International Journal of Machine Learning and Cybernetics10,1513–1528.doi:10.1007/s13042-018-0830-9 1

[25] Xu, Z. (2006). Incomplete linguistic preference relations and their fusion. Information Fusion, 7(3), 331–337. doi:10.1016/j.inffus.2005.01.003 1

[26] Xie, W., Xu, Z., Ren, Z., & Herrera-Viedma, E. (2020). A new multi-criteria decision model based on incomplete dual probabilistic linguistic preference relations. Applied Soft Computing, 106237. doi:10.1016/j.asoc.2020.106237 1

[27] Zhang, Y., Xu, Z., Wang, H., & Liao, H. (2016). Consistency-based risk assessment with probabilistic linguistic preference relation. Applied Soft Computing, 49, 817– 833. doi:10.1016/j.asoc.2016.08.045 1

[28] Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8, 338-353.doi: 10.1016/50019-9958(65)90241-X 1, 2.1, 2.2

[29] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning-III. Information Sciences, 9(1), 43–80. doi:10.1016/0020- 0255(75)90017-1 1

[30] Zadeh, L.A. (1971). Similarity relations and fuzzy orderings. , 3(2), 177–200. doi:10.1016/s0020-0255(71)80005-1
1