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Generalized Integral Inequalities Involving Some Special Functions with means Application

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Abstract

Several novel generalized integral inequalities for $(\xi, s, m_1)-$convex functions are established in this study. Trapezoid, Bullen, and Simpson's like inequalities for $(\xi, s, m_1)-$convexïty were obtained as specific examples. By utilizing conventional inequalities like Hölder inequality, we have used a very traditional methodology.Improved Power-mean inequality and Ican inequality. We provide a number of constraints involving special functions, such as the $\Psi-$Gamma(Di-gamma functions), Beta, and classical Euler-Gamma functions. Lastly, we also provide various means application.

Keywords: convexity, $(\xi, s, m_1)-$convex function, Hölder's integral inequality, Power mean integral inequality, Hölder-İşcan integral inequality, Improved power-mean inequality, Beta function, Gamma function, di-gamma function.

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1. Introduction

A study of convex sets and geometry in the 19th century gave rise to the idea of convex functions. Mathematicians like Augustin-Louis Cauchy and Hermann Minkowski, who investigated convex shapes and inequalities in geometry, are credited with developing early concepts linked to convexity. The development of convex analysis, which linked geometry, calculus, and optimization theory, led to the formal formulation of convex functions (see [1]-[9]). Convex functions rose to prominence in a number of mathematical domains during the 20th century, including economics, optimization, and functional analysis. Contributions to the field were made by scholars such as Albert W. Tucker and Harold W. Kuhn, who introduced the renowned Kuhn-Tucker conditions and convex optimization. Convex functions are now the cornerstone of contemporary optimization theory and are essential to fields like economics, machine learning, and operations research.

Convexity of the theory has acted as an influential practice to study a different type of class with distinct issues in applied and pure literature of the natural sciences. Several courses and articles have been printed by numeral of mathematicians on convex functions and inequalities for their altered classes, using, e.g, the articles (see [10]-[14]) and the references therein. There are some basic and vital definitions

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related to the idea of convexity of a functions.

The following definition can be seen in the different related literature (see [13]).

Definition 1.1. A mapping $\mathfrak{T} : \mathfrak{J} \subseteq \mathfrak{R} \to \mathfrak{R}$ is called convexity, if

$$\mathfrak{T}(\mathfrak{V}\flat_3 + (1-\mathfrak{V})\flat_4) \leqslant \mathfrak{V}\mathfrak{T}(\flat_3) + (1-\mathfrak{V})\mathfrak{T}(\flat_4)$$

$\forall\ \flat_3, \flat_4 \in \mathfrak{J}$ and $\mathfrak{V} \in [\,0,1]$.

It will be concaved, if above is reversed.

Definition 1.2 ([7])**.** A mapping $\mathfrak{T} : [0,c] \to \mathfrak{R}_0$=$[0,\infty[$ is called $\mathfrak{m}_1-$convexity for some $\mathfrak{m}_1 \in]0,1]$, $c > 0$, if we have

$$\mathfrak{T}(\mathfrak{V}\flat_3 + \mathfrak{m}_1(1-\mathfrak{V})\flat_4) \leqslant \mathfrak{V}\mathfrak{T}(\flat_3) + \mathfrak{m}_1(1-\mathfrak{V})\mathfrak{T}(\flat_4)$$

for all $\flat_3, \flat_4 \in [0,c]$ and $\mathfrak{V} \in [0,1]$.

Definition 1.3 ([8])**.** A mapping $\mathfrak{T} : [0,c] \to \mathfrak{R}_0$ is told to be $(\xi, \mathfrak{m}_1)-$convexity for $(\xi, \mathfrak{m}_1) \in]0,1]^2$, $c > 0$, if

$$\mathfrak{T}(\mathfrak{V}\flat_3 + \mathfrak{m}_1(1-\mathfrak{V})\flat_4) \leqslant \mathfrak{V}^{\xi}\mathfrak{T}(\flat_3) + \mathfrak{m}_1(1-\mathfrak{V}^{\xi})\mathfrak{T}(\flat_4)$$

$\forall\ \flat_3, \flat_4 \in [0,c]$ and $\mathfrak{V} \in [0,1]$.

Suppose $K_{\xi}^{\mathfrak{m}_1}$ is the set of $(\xi, \mathfrak{m}_1)-$convexity on $[0,c]$.

Definition 1.4 ([9])**.** A mapping $\mathfrak{T} : \mathfrak{J} \subseteq \mathfrak{R} \to \mathfrak{R}_0$ is told to be $s-$convexity (in the 2^{nd}-type sense) if

$$\mathfrak{T}(\mathfrak{V}\flat_3 + (1-\mathfrak{V})\flat_4) \leqslant \mathfrak{V}^s\mathfrak{T}(\flat_3) + (1-\mathfrak{V})^s\mathfrak{T}(\flat_4)$$

$\forall\ \flat_3, \flat_4 \in \mathfrak{J}$, $\mathfrak{V} \in [0,1]$ and for some $s \in]0,1]$.

Definition 1.5 ([10])**.** A mapping $\mathfrak{T} : \mathfrak{J} \subseteq \mathfrak{R}^+ \to \mathfrak{R}$ is called extendable $s-$convexity, if

$$\mathfrak{T}(\mathfrak{V}\flat_3 + (1-\mathfrak{V})\flat_4) \leqslant \mathfrak{V}^s\mathfrak{T}(\flat_3) + (1-\mathfrak{V})^s\mathfrak{T}(\flat_4)$$

$\forall\ \flat_3, \flat_4 \in \mathfrak{J}$, $\mathfrak{V} \in]0,1[$ and for some $s \in [-1,1]$.

Definition 1.6 ([11])**.** A mapping $\mathfrak{T} : [0,c] \to \mathfrak{R}_0$ is told to be extendable $(s, \mathfrak{m}_1)-$convexity, if

$$\mathfrak{T}(\mathfrak{V}\flat_3 + \mathfrak{m}_1(1-\mathfrak{V})\flat_4) \leqslant \mathfrak{V}^s\mathfrak{T}(\flat_3) + \mathfrak{m}_1(1-\mathfrak{V})^s\mathfrak{T}(\flat_4)$$

$\forall\ \flat_3, \flat_4 \in [0,c]$, $\mathfrak{V} \in]0,1[$ and for some $(s, \mathfrak{m}_1) \in [-1,1] \times]0,1]$.

The Hermite-Hadamard inequality has been extendable, generalized, counterparted, and improved for several classes of convex maps by the mathematicians working in computational and applied area mathematics. We refer the reader to References (see [7]-[10]).

Numerous writers have demonstrated a number of inequalities or inequalities for convexity, as the Ostrowski like inequality, Hardy type inequality, and Cebysev type inequalities; however, the most prevalent and significant inequality is the Hermite-Hadamard type inequality (see [16]- [21]), which is defined as:

Theorem 1.7. *A mapping $\mathfrak{T} : \mathfrak{J} \subseteq \mathfrak{R} \to \mathfrak{R}$ is a convexity in $\mathfrak{J}$ and $\flat_3, \flat_4 \in \mathfrak{J}$, $\flat_3 < \flat_4$, then*

$$\mathfrak{T}\left(\frac{\flat_3 + \flat_4}{2}\right) \leqslant \frac{1}{\flat_4 - \flat_3} \int_{\flat_3}^{\flat_4} \mathfrak{T}(\mathfrak{V})\, d\mathfrak{V} \leqslant \frac{\mathfrak{T}(\flat_3) + \mathfrak{T}(\flat_4)}{2}.$$

The Hermite–Hadamard inequality, dating back to 1883, is often viewed as one of the highly practically significant inequalities in the field of analysis in mathematical. Another name for it is the classical form of the H-H inequality.

Theorem 1.8. *Suppose a mapping $\mathfrak{T} : \mathfrak{J} \subseteq \mathfrak{R} \to \mathfrak{R}$ is differentiable in $\mathfrak{J}^\circ$, $\flat_3, \flat_4 \in \mathfrak{J}^\circ$, $\flat_3 < \flat_4$. If $|\mathfrak{T}'|$ is convex on the $[\flat_3, \flat_4]$ closed interval, then*

$$\left|\frac{\mathfrak{T}(\flat_3)+\mathfrak{T}(\flat_4)}{2} - \frac{1}{\flat_4-\flat_3}\int_{\flat_3}^{\flat_4} \mathfrak{T}(\mathfrak{V})\,d\mathfrak{V}\right| \leqslant \frac{(\flat_4-\flat_3)\left(|\mathfrak{T}'(\flat_3)|+|\mathfrak{T}'(\flat_4)|\right)}{8}.$$

In this paper, we discuss some ideas about $(\xi, s, \mathfrak{m}_1)-$convexity and develop several inequalities related to Hadamard- type for such mapping whose two times derivative is $(\xi, s, \mathfrak{m}_1)-$convex.

2. Some Important Definitions

Now we develop some new idea related to (ξ, s) and $(\xi, s, \mathfrak{m}_1)-$convexity.

Definition 2.1. [12] Consider a transformation $\mathfrak{T} : \mathfrak{J} \subseteq \mathfrak{R} \to \mathfrak{R}$ is called $(\xi, s)-$convexity, if we have

$$\mathfrak{T}(\mathfrak{V}\flat_3 + (1-\mathfrak{V})\flat_4) \leqslant \mathfrak{V}^{\xi,s}\mathfrak{T}(\flat_3) + (1-\mathfrak{V}^{\xi})^s\mathfrak{T}(\flat_4)$$

$\forall\ \flat_3, \flat_4 \in \mathfrak{T}$ and $\mathfrak{V} \in (0, 1)$ and for some $s \in [-1, 1]$ and $\xi \in]0, 1]$.

Definition 2.2. [12, 13] Consider a transformation $\mathfrak{T} : [0, c] \to \mathfrak{R}$ is told to be $(\xi, s, \mathfrak{m}_1)-$convexity, if we have

$$\mathfrak{T}(\mathfrak{V}\flat_3 + \mathfrak{m}_1(1-\mathfrak{V})\flat_4) \leqslant \mathfrak{V}^{\xi,s}\mathfrak{T}(\flat_3) + \mathfrak{m}_1(1-\mathfrak{V}^{\xi})^s\mathfrak{T}(\flat_4)$$

$\forall\ \flat_3, \flat_4 \in [0, c]$, $\mathfrak{V} \in]0, 1[$ and for some $s \in [-1, 1]$ and $(\xi, \mathfrak{m}_1) \in]0, 1]^2$.

Here $K_{\xi}^{s,\mathfrak{m}_1}$ is the set of $(\xi, s, \mathfrak{m}_1)-$convexity on $[0, c]$.

Some Important Note :

1. Consider a transformation $\mathfrak{T}$ is an $(\xi, \mathfrak{m}_1)-$convexity in $]0, c]$, if we take $s = 1$.
2. Consider a transformation $\mathfrak{T}$ is extendable $(s, \mathfrak{m}_1)-$convexity in $]0, c]$, if we take $1 = \xi$.
3. The mapping $\mathfrak{T}$ is extended $s-$convexity in $]0, c]$, if we take $\mathfrak{m}_1 = 1 = \xi$.
4. The $\mathfrak{T}$ is a convexity in $]0, c]$, if we take $\xi = \mathfrak{m}_1 = s = 1$.

Remark 2.3. Let $\xi \in (0, 1]$. Then

1. If $s \in]-1, 1]$, we have

$$U(\xi, s) = \int_0^1 \mathfrak{V}(1-\mathfrak{V})(1-\mathfrak{V}^{\xi})^s\,d\mathfrak{V} = \frac{1}{\xi}\left[\beta\left(\frac{2}{\xi}, s+1\right) - \beta\left(\frac{3}{\xi}, s+1\right)\right]. \tag{2.1}$$

2. If $s = -1$, we get

$$\int_0^1 \frac{\mathfrak{V}(1-\mathfrak{V})}{1-\mathfrak{V}^{\xi}}\,d\mathfrak{V} = \frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right) - \Psi\left(\frac{2}{\xi}\right)\right]. \tag{2.2}$$

3. Now we use the fact that $\mathfrak{V}^{\xi} \geqslant \mathfrak{V}, \forall\ \xi \in (0, 1]$ and $\mathfrak{V} \in$ [0,1]

$$\left(1-\mathfrak{V}^{\xi}\right)^s \leqslant (1-\mathfrak{V})^{\xi,s}\ ; \text{for some } s \in (0, 1].$$

We multiplying the term $\mathfrak{V}(1-\mathfrak{V})$ and taking integral on the $[0, 1]$, we have

$$\int_0^1 \mathfrak{V}\quad (1-\mathfrak{V})\left(1-\ \mathfrak{V}^{\ \xi}\right)^s \quad \mathfrak{V} \qquad \leqslant \qquad \int_0^1 \mathfrak{V}\,(1-\ \mathfrak{V})\,(1-\mathfrak{V})^{\xi,s} \quad \mathfrak{V} \quad = \quad \frac{1}{(\xi s+2)\,(\xi s+3)}.$$

See the following literature in [13], E-gamma and Di-gamma functions as described by

$$\Gamma(u) = \int_0^1 \mathfrak{V}^{u-1} e^{-\mathfrak{V}} d\mathfrak{V}, \quad \beta(u,v) = \int_0^1 \mathfrak{V}^{u-1}(1-\mathfrak{V})^{v-1} d\mathfrak{V} \tag{2.3}$$

and

$$\Psi(u) = \frac{d \ln \Gamma(u)}{du} = \int_0^\infty \left[\frac{e^{-\mathfrak{V}}}{\mathfrak{V}} - \frac{e^{-u\mathfrak{V}}}{1-e^{-\mathfrak{V}}}\right] d\mathfrak{V} \tag{2.4}$$

for $u, v > 0$.

Proof. Consider that $u = \mathfrak{V}^{\xi}$ for $\mathfrak{V} \in [0,1]$. If $s \in (-1,1]$, we have

$$\begin{aligned}
&\int_0^1 \mathfrak{V}(1-\mathfrak{V})(1-\mathfrak{V}^{\xi})^s \mathfrak{V} \\
&= \frac{1}{\xi}\int_0^1 \left(u^{2/\xi-1} - u^{3/\xi-1}\right)(1-u)^s u \\
&= \frac{1}{\xi}\left[\beta\left(s+1, \frac{2}{\xi}\right) - \beta\left(s+1, \frac{3}{\xi}\right)\right].
\end{aligned}$$

where $s = -1$, from the formulas

$$\Psi(z) + \gamma = \int_0^1 \frac{1-\mathfrak{V}^{z-1}}{1-\mathfrak{V}} d\mathfrak{V}, \quad \text{and} \quad \gamma = \int_0^\infty \left(\frac{1}{1+\mathfrak{V}} - e^{-\mathfrak{V}}\right)\frac{d\mathfrak{V}}{\mathfrak{V}}.$$

It can be easily to find

$$\int_0^1 \frac{\mathfrak{V}(1-\mathfrak{V})}{1-\mathfrak{V}^{\xi}} d\mathfrak{V} = \frac{1}{\xi}\int_0^1 \frac{u^{2/\xi-1} - u^{3/\xi-1}}{1-u} du = \frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right) - \Psi\left(\frac{2}{\xi}\right)\right], \tag{2.5}$$

which is completed. □

Throughout the paper we consider $\mathfrak{T}^{\circ}$ is the interior of $\mathfrak{T}$.

3. Main Results

Firstly, we proposed the modification of the Lemma 1 in [14] and then We gave the new integral inequalities of the Hadamard type by pointing $(\xi, s, m_1)-$convexity, so we require Lemma as defined below:

Lemma 3.1. *Suppose a mapping $\mathfrak{T} : \mathfrak{J} \subseteq \mathfrak{R} \to \mathfrak{R}$ is two times differentiable on $\mathfrak{J}^{\circ}$ of an interval $\mathfrak{J}$ in $\mathfrak{R}$, where $\flat_3, \flat_4 \in \mathfrak{J}$, $\flat_3 < \flat_4$. If $\mathfrak{T}'' \in L_1[\flat_3, \flat_4]$, then for $\mathfrak{V} \in [0,1]$ and $n \geqslant 1$, we have the following generalized identity*

$$\begin{aligned}
&\frac{1}{2n}\left[\mathfrak{T}(\flat_3) + (n-1)\left\{\mathfrak{T}\left(\frac{\flat_3 + (n-1)\flat_4}{n}\right) + \mathfrak{T}\left(\frac{(n-1)\flat_3 + \flat_4}{n}\right)\right\} + \mathfrak{T}(\flat_4)\right] \\
&\qquad - \frac{1}{\flat_4 - \flat_3}\int_{\flat_3}^{\flat_4} \mathfrak{T}(\mathfrak{V})\, d\mathfrak{V} \\
&= \frac{(\flat_4 - \flat_3)^2}{2n^3}\left[\int_0^1 \mathfrak{V}(1-\mathfrak{V})\ \mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3 + \flat_4}{n} + (1-\mathfrak{V})\flat_3\right) d\mathfrak{V}\right. \\
&\quad + \int_0^1 \mathfrak{V}(1-\mathfrak{V})\ \mathfrak{T}''\left(\mathfrak{V}\flat_4 + (1-\mathfrak{V})\frac{\flat_3 + (n-1)\flat_4}{n}\right) d\mathfrak{V} \\
&\quad \left. + (n-2)^3 \int_0^1 \mathfrak{V}(1-\mathfrak{V})\ \mathfrak{T}''\left(\mathfrak{V}\frac{\flat_3 + (n-1)\flat_4}{n} + (1-\mathfrak{V})\frac{(n-1)\flat_3 + \flat_4}{n}\right) d\mathfrak{V}\right].
\end{aligned} \tag{3.1}$$

Proof. It suffices to note that

$$\frac{(\flat_4-\flat_3)^2}{2n^3}\Bigg[\int_0^1 \mathfrak{V}(1-\mathfrak{V})\ \mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+(1-\mathfrak{V})\flat_3\right)d\mathfrak{V}$$
$$+\int_0^1 \mathfrak{V}(1-\mathfrak{V})\ \mathfrak{T}''\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\frac{\flat_3+(n-1)\flat_4}{n}\right)d\mathfrak{V}$$
$$+(n-2)^3\int_0^1 \mathfrak{V}(1-\mathfrak{V})\ \ \mathfrak{T}''\left(\mathfrak{V}\frac{\flat_3+(n-1)\flat_4}{n}+(1-\mathfrak{V})\frac{(n-1)\flat_3+\flat_4}{n}\right)d\mathfrak{V}\Bigg]$$

$$= \frac{(\flat_4-\flat_3)^2}{2n^3}\left[\{I_1+I_2\}+(n-2)^3 I_3\right]. \tag{3.2}$$

$$I_1 = \int_0^1 \mathfrak{V}(1-\mathfrak{V})\mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+(1-\mathfrak{V})\flat_3\right)d\mathfrak{V}$$

(Ibp)Integrating by parts, we have

$$= -\frac{n}{\flat_4-\flat_3}\int_0^1 (1-2\mathfrak{V})\mathfrak{T}'\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+(1-\mathfrak{V})\flat_3\right)d\mathfrak{V}$$

$$= \frac{n^2}{(\flat_4-\flat_3)^2}\left\{\mathfrak{T}(\flat_3)\ +\mathfrak{T}\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right\} - \frac{2n^3}{(\flat_4-\flat_3)^3}\int_{\flat_3}^{\frac{(n-1)\flat_3+\flat_4}{n}} \mathfrak{T}(\mathfrak{V})\ \ d\mathfrak{V},$$

and

$$I_2 = \int_0^1 (1-\mathfrak{V})\mathfrak{V}\ \ \ \mathfrak{T}''\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\frac{\flat_3+(n-1)\flat_4}{n}\right)d\mathfrak{V}$$

Ibp, we have

$$= -\frac{n}{\flat_4-\flat_3}\int_0^1 (1-2\mathfrak{V})\mathfrak{T}'\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\frac{\flat_3+(n-1)\flat_4}{n}\right)d\mathfrak{V}$$

$$= \frac{n^2}{(\flat_4-\flat_3)^2}\left\{\mathfrak{T}(\flat_4)\ +\mathfrak{T}\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right\} - \frac{2n^3}{(\flat_4-\flat_3)^3}\int_{\frac{\flat_3+(n-1)\flat_4}{n}}^{\flat_4} \mathfrak{T}(\mathfrak{V})\,d\mathfrak{V}$$

and

$$I_3 = \int_0^1 \mathfrak{V}\ \ \ (1-\mathfrak{V})\mathfrak{T}''\left(\mathfrak{V}\frac{\flat_3+(n-1)\flat_4}{n}+\frac{(n-1)\flat_3+\flat_4}{n}(1-\mathfrak{V})\right)d\mathfrak{V}$$

Integrating by parts, we find

$$= -\frac{n}{(n-2)(\flat_4-\flat_3)}\int_0^1 (1-2\mathfrak{V})\mathfrak{T}'\left(\mathfrak{V}\frac{\flat_3+(n-1)\flat_4}{n}+\frac{(n-1)\flat_3+\flat_4}{n}(1-\mathfrak{V})\right)d\mathfrak{V}$$

$$= \frac{n^2}{(n-2)^2(\flat_4-\flat_3)^2}\left\{\mathfrak{T}\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)+\mathfrak{T}\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right\} - \frac{2n^3}{(n-2)^3(\flat_4-\flat_3)^3}\int_{\frac{(n-1)\flat_3+\flat_4}{n}}^{\frac{\flat_3+(n-1)\flat_4}{n}}\mathfrak{T}(\mathfrak{V})\ d\mathfrak{V},$$

combining I_1, I_2 and I_3 with (3.2) and we get (3.1). □

As a case for n, we have the Remarks as below:

Remark 3.2. When $n=1$, the equality (3.1) reduces to

$$\frac{\mathfrak{T}(\flat_3)+\mathfrak{T}(\flat_4)}{2}-\frac{1}{\flat_4-\flat_3}\int_{\flat_3}^{\flat_4}\mathfrak{T}(x)\ dx = \frac{(\flat_4-\flat_3)^2}{2}\left[2\int_0^1\mathfrak{V}(1-\mathfrak{V})\mathfrak{T}''(\mathfrak{V}\flat_4+(1-\mathfrak{V})\flat_3)\ d\mathfrak{V} - \int_0^1\mathfrak{V}(1-\mathfrak{V})\mathfrak{T}''(\mathfrak{V}\flat_3+(1-\mathfrak{V})\flat_4)\ d\mathfrak{V}\right].$$

Remark 3.3. When $n=2$, the equality (3.1) reduces to

$$\frac{1}{4}\left[\mathfrak{T}(\flat_3)+2\mathfrak{T}\left(\frac{\flat_3+\flat_4}{2}\right)+\mathfrak{T}(\flat_4)\right]-\frac{1}{\flat_4-\flat_3}\int_{\flat_3}^{\flat_4}\mathfrak{T}(x)\ dx$$
$$=\frac{(\flat_4-\flat_3)^2}{16}\left[\int_0^1\mathfrak{V}(1-\mathfrak{V})\left\{\mathfrak{T}''\left(\mathfrak{V}\frac{\flat_3+\flat_4}{2}+(1-\mathfrak{V})\flat_3\right)+\mathfrak{T}''\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\frac{\flat_3+\flat_4}{2}\right)\right\}d\mathfrak{V}\right].$$

The equality in Remark 3.3 is proved in (Lemma 2.1, [12]).

Now we generate some new inequalities related to Hadamard types integral for (ξ, s, m_1)−convexity.

Theorem 3.4. *Suppose a mapping* $\mathfrak{T}: \mathfrak{J}\subset (0, \frac{c^*}{m_1}] \to \mathfrak{R}$ *is two times differentiable on* $\mathfrak{J}^\circ$ *of the interval* $\mathfrak{J}$ *in* $\mathfrak{R}$, *where* $\flat_3, \flat_4 \in (0, c^*]$ *with* $\flat_3 < \flat_4$. *If* $\mathfrak{T}'' \in L_1[\flat_3, \flat_4]$ and $|\mathfrak{T}''|$ *is* (ξ, s, m_1)−*convexity on* $(0, \frac{c^*}{m_1}]$ *for* $(\xi, m_1) \in]0,1]^2$ *and* $s\in(-1,1]$, *then below holds:*

$$|M(\mathfrak{T}; n, \flat_3, \flat_4)| \leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\left[\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right| + m_1\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|+|\mathfrak{T}''(\flat_4)|+m_1\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|}{(\xi s+2)(\xi s+3)}\right\} + (n-2)^3\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|+m_1\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|}{(\xi s+2)(\xi s+3)}\right\}\right], \quad (3.3)$$

where $M(\mathfrak{T}; n, \flat_3, \flat_4)$ *is the left hand side of Lemma 3.1.*

Proof. With equation (3.1) and using the (ξ, s, m_1)−convexity of $|\mathfrak{T}''|$ and the property of modulus, we obtain

(3.4)

$$
\begin{aligned}
&|M(\mathfrak{T};n,\flat_3,\flat_4)| \\
&\quad\leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\left[\left\{\int_0^1 \mathfrak{V}(1-\mathfrak{V})\left(\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|\right)d\mathfrak{V}\right.\right.\\
&\qquad\left.+\int_0^1 \mathfrak{V}(1-\mathfrak{V})\left(\mathfrak{V}^{\xi s}\left|\mathfrak{T}''(\flat_4)\right|+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{\mathfrak{m}_1 n}\right)\right|\right)d\mathfrak{V}\right\}\\
&\quad\left.+(n-2)^3\int_0^1 \mathfrak{V}(1-\mathfrak{V})\left(\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{\mathfrak{m}_1 n}\right)\right|\right)d\mathfrak{V}\right].
\end{aligned}
$$

Employing Remark 2.3 and using the calculas tools, we get (3.3).

□

The obtained general integral inequality gives some renowned inequalities by substituting particular values of n.

Remark 3.5. (a) For $n=1$ in (3.3), it becomes Trapezoid type inequality

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\mathfrak{T}(\flat_3)+\mathfrak{T}(\flat_4)\right]-\frac{1}{\flat_4-\flat_3}\int_{\flat_3}^{\flat_4}\mathfrak{T}(\mathfrak{V})\,d\mathfrak{V}\right|\\
&\qquad\leqslant \frac{(\flat_4-\flat_3)^2}{2}\left[\left\{\frac{2\left|\mathfrak{T}''(\flat_4)\right|+2\mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|}{(\xi s+2)(\xi s+3)}\right\}-\left\{\frac{\left|\mathfrak{T}''(\flat_3)\right|+\mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{\flat_4}{\mathfrak{m}_1}\right)\right|}{(\xi s+2)(\xi s+3)}\right\}\right].
\end{aligned}
$$

(b) For $n=2$ in (3.3), it becomes Bullen type inequality

$$
\begin{aligned}
&\left|\frac{1}{4}\left[\mathfrak{T}(\flat_3)+2\mathfrak{T}\left(\frac{\flat_3+\flat_4}{2}\right)+\mathfrak{T}(\flat_4)\right]-\frac{1}{\flat_4-\flat_3}\int_{\flat_3}^{\flat_4}\mathfrak{T}(\mathfrak{V})\,d\mathfrak{V}\right|\\
&\qquad\leqslant \frac{(\flat_4-\flat_3)^2}{16}\left[\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+\flat_4}{2}\right)\right|+\mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|+\left|\mathfrak{T}''(\flat_4)\right|+\mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{\flat_3+\flat_4}{2\mathfrak{m}_1}\right)\right|}{(\xi s+2)(\xi s+3)}\right\}\right].
\end{aligned}
$$

(c) For $n=3$ in (3.3), it becomes Simpson type inequality

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\mathfrak{T}(\flat_3)+2\left\{\mathfrak{T}\left(\frac{\flat_3+2\flat_4}{3}\right)+\mathfrak{T}\left(\frac{2\flat_3+\flat_4}{3}\right)\right\}+\mathfrak{T}(\flat_4)\right]-\frac{1}{\flat_4-\flat_3}\int_{\flat_3}^{\flat_4}\mathfrak{T}(\mathfrak{V})\,d\mathfrak{V}\right|\\
&\qquad\leqslant \frac{(\flat_4-\flat_3)^2}{54}\\
&\qquad\times\left[\left\{\frac{\left|\mathfrak{T}''\left(\frac{2\flat_3+\flat_4}{3}\right)\right|+\mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|+\left|\mathfrak{T}''(\flat_4)\right|+\mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{\flat_3+2\flat_4}{3\mathfrak{m}_1}\right)\right|}{(\xi s+2)(\xi s+3)}\right\}\right.\\
&\qquad\left.+\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+2\flat_4}{3}\right)\right|+\mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{2\flat_3+\flat_4}{3\mathfrak{m}_1}\right)\right|}{(\xi s+2)(\xi s+3)}\right\}\right].
\end{aligned}
$$

Corollary 3.6. *Suppose a mapping* $\mathfrak{T}:\mathfrak{J}\subset(0,c^*]\subseteq\mathfrak{R}\to\mathfrak{R}$ *is two times differentiable and also that* $\mathfrak{T}''\in L_1([\flat_3,\flat_4])$ *for* $\flat_3,\flat_4\in J$ *with* $\flat_3<\flat_4$. *If* $|\mathfrak{T}''|$ *is* $(\xi,s)-$*convex function on set* $\mathfrak{J}$ *and for* $\xi\in]0,1]$ *and* $s\in(-1,1]$, *then*

$$|M(\mathfrak{T};n,\flat_3,\flat_4)| \leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\left[\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|+|\mathfrak{T}''(\flat_3)|+|\mathfrak{T}''(\flat_4)|+\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|}{(\xi s+2)(\xi s+3)}\right\} + (n-2)^3\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|+\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|}{(\xi s+2)(\xi s+3)}\right\}\right]. \quad (3.5)$$

Proof. With a case of Theorem 3.4 by putting $m_1 = 1$. □

Theorem 3.7. *Suppose a mapping $\mathfrak{T}:\left(0,\frac{c^*}{m_1}\right]\to\mathfrak{R}$ is two times differentiable on $\mathfrak{J}^\circ$ and $\mathfrak{T}''\in L_1([\flat_3,\flat_4])$ for $\flat_3,\flat_4\in(0,c^*]$ with $\flat_3<\flat_4$. If $|\mathfrak{T}''|^q$ is (ξ,s,m_1)−convexity on $\left(0,\frac{c^*}{m_1}\right]$, $q\geqslant 1$, for some $(\xi,m_1)\in(0,1]^2$ and $s\in(-1,1]$, then below holds:*

$$\begin{aligned}|M(\mathfrak{T};n,\flat_3,\flat_4)| &\leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\times\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+2)(\xi s+3)}+m_1.U(\xi,s)\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|^q\right\}^{\frac{1}{q}}\right.\\ &+\left\{\frac{|\mathfrak{T}''(\flat_4)|^q}{(\xi s+2)(\xi s+3)}+m_1U(\xi,s)\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q\right\}^{\frac{1}{q}}\\ &\left.+(n-2)^3\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+2)(\xi s+3)}+m_1U(\xi,s)\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q\right\}^{\frac{1}{q}}\right], \quad (3.6)\end{aligned}$$

where $U(\xi,s)$ is defined in Remark 2.3, and $M(\mathfrak{T};n,\flat_3,\flat_4)$ denotes the left-hand side of Lemma 3.1.

Proof. Firstly, we consider the case for $q=1$. Now using equation (3.1) with (ξ,s,m_1)−convexity of $|\mathfrak{T}''|$ and property of modulus, we have

$$\begin{aligned}|M(\mathfrak{T};n,\flat_3,\flat_4)| &\leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\left[\left\{\int_0^1\mathfrak{V}(1-\mathfrak{V})\left(\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|+m_1(1-\mathfrak{V}^\xi)^s\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|\right)d\mathfrak{V}\right.\right.\\ &\left.+\int_0^1\mathfrak{V}(1-\mathfrak{V})\left(\mathfrak{V}^{\xi s}|\mathfrak{T}''(\flat_4)|+m_1(1-\mathfrak{V}^\xi)^s\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|\right)d\mathfrak{V}\right\}\\ &+(n-2)^3\int_0^1\mathfrak{V}(1-\mathfrak{V})\\ &\left.\times\left(\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|+m_1(1-\mathfrak{V}^\xi)^s\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|\right)d\mathfrak{V}\right]. \quad (3.7)\end{aligned}$$

Now simplifying (3.7), we get

$$
\begin{aligned}
&\leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\left[\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|}{(\xi s+2)(\xi s+3)}+m_1 U(\xi,s)\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|\right\}\right.\\
&+\left\{\frac{|\mathfrak{T}''(\flat_4)|}{(\xi s+2)(\xi s+3)}+m_1 U(\xi,s)\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|\right\}\\
&\left.+(n-2)^3\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|}{(\xi s+2)(\xi s+3)}+ m_1 U(\xi,s)\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|\right\}\right] \qquad (3.8)
\end{aligned}
$$

and the proof is completed for $q=1$ by using Remark 2.3.

Now we consider the case $q>1$. With equation (3.1) and using the power-mean inequality for $q>1$, with modulus properties, we get

$$
\begin{aligned}
&\int_0^1 \mathfrak{V}(1-\mathfrak{V})\left|\mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+(1-\mathfrak{V})\flat_3\right)\right| d\mathfrak{V}\\
&=\int_0^1 (\mathfrak{V}-\mathfrak{V}^2)^{1-\frac{1}{q}}(\mathfrak{V}-\mathfrak{V}^2)^{\frac{1}{q}}\left|g''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+m_1(1-\mathfrak{V})\left(\frac{\flat_3}{m_1}\right)\right)\right| d\mathfrak{V}\\
&\leqslant \left[\int_0^1 (\mathfrak{V}-\mathfrak{V}^2)\ d\mathfrak{V}\right]^{1-\frac{1}{q}}\left[\int_0^1 (\mathfrak{V}-\mathfrak{V}^2)\left|\mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+m_1(1-\mathfrak{V})\left(\frac{\flat_3}{m_1}\right)\right)\right|^q d\mathfrak{V}\right]^{\frac{1}{q}},
\end{aligned}
$$

since $|\mathfrak{T}''|^q$ is $(\xi,s,m_1)-$convex on $(0,\frac{c^*}{m_1}]$, we know that for every $\mathfrak{V}\in[0,1]$

$$
\left|\mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+m_1(1-\mathfrak{V})\left(\frac{\flat_4}{m_1}\right)\right)\right|^q \leqslant \mathfrak{V}^{\xi,s}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q+m_1(1-\mathfrak{V}^{\xi})^s\left|\mathfrak{T}''\left(\frac{\flat_4}{m_1}\right)\right|^q,
$$

then we have

$$
\begin{aligned}
|M(\mathfrak{T};n,\flat_3,\flat_4)| &\leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\left[\left\{\int_0^1(\mathfrak{V}-\mathfrak{V}^2)\ d\mathfrak{V}\right\}^{1-\frac{1}{q}}\right.\\
&\times\left\{\int_0^1(\mathfrak{V}-\mathfrak{V}^2)\left(\mathfrak{V}^{\xi,s}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q+m_1(1-\mathfrak{V}^{\xi})^s\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|^q\right)d\mathfrak{V}\right\}^{\frac{1}{q}}\\
&+\left\{\int_0^1(\mathfrak{V}-\mathfrak{V}^2)\ d\mathfrak{V}\right\}^{1-\frac{1}{q}}\\
&\times\left\{\int_0^1(\mathfrak{V}-\mathfrak{V}^2)\left(\mathfrak{V}^{\xi,s}|\mathfrak{T}''(\flat_4)|^q+m_1(1-\mathfrak{V}^{\xi})^s\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q\right)d\mathfrak{V}\right\}^{\frac{1}{q}}\\
&+(n-2)^3\left\{\int_0^1(\mathfrak{V}-\mathfrak{V}^2)\ d\mathfrak{V}\right\}^{1-\frac{1}{q}}\\
&\left.\times\left\{\int_0^1(\mathfrak{V}-\mathfrak{V}^2)\left(\mathfrak{V}^{\xi,s}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q+m_1(1-\mathfrak{V}^{\xi})^s\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q\right)d\mathfrak{V}\right\}^{\frac{1}{q}}\right]. \qquad (3.9)
\end{aligned}
$$

By using the calculas tools, one can have the eq. (3.6). □

Corollary 3.8. *Suppose a mapping $\mathfrak{T}:\mathfrak{J}\subseteq\mathfrak{R}\to\mathfrak{R}$ is two times differentiable and $\mathfrak{T}''\in L_1([\flat_3,\flat_4])$ for $\flat_3,\flat_4\in\mathfrak{J}$ with $\flat_3<\flat_4$. If $|\mathfrak{T}''|^q$ is an $(\xi,s)-$convexity on $\mathfrak{J}$ for $\xi\in]0,1]$, $s\in(-1,1]$ and $q\geqslant 1$, then*

$$
\begin{aligned}
|M(\mathfrak{T};n,\flat_3,\flat_4)| \leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\times\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\Bigg[&\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+2)(\xi s+3)}+U(\xi,s)\left|\mathfrak{T}''(\flat_3)\right|^q\right\}^{\frac{1}{q}}\\
&+\left\{\frac{|\mathfrak{T}''(\flat_4)|^q}{(\xi s+2)(\xi s+3)}+U(\xi,s)\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q\right\}^{\frac{1}{q}}\\
&+(n-2)^3\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+2)(\xi s+3)}+U(\xi,s)\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q\right\}^{\frac{1}{q}}\Bigg], \quad (3.10)
\end{aligned}
$$

where $U(\xi, s)$ *is clearly defined in* (2.3).

Proof. This case is related for Theorem 3.7 by taking $m=1$. □

Theorem 3.9. *Suppose a mapping* $\mathfrak{T}:\left(0,\frac{c^*}{\mathfrak{m}_1}\right]\to\mathfrak{R}$ *is two times differentiable and* $\mathfrak{T}''\in L_1([\flat_3,\flat_4])$ *for* $\flat_3,\flat_4\in(0,c^*]$ *with* $\flat_3<\flat_4$. *If* $|\mathfrak{T}''|^q$ *is* $(\xi,-1,\mathfrak{m}_1)$*–convexity on* $\left(0,\frac{c^*}{\mathfrak{m}_1}\right]$, $q\geqslant 1$, *for some* $(\xi,\mathfrak{m}_1)\in(0,1]^2$, *then the following inequality holds:*

$$
\begin{aligned}
&|M(\mathfrak{T};n,\flat_3,\flat_4)|\\
&\leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\times\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\Bigg[\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_3}{n}\right)\right|^q}{(2-\xi)(3-\xi)}+\frac{\mathfrak{m}_1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|^q\right\}^{\frac{1}{q}}\\
&\quad+\left\{\frac{|\mathfrak{T}''(\flat_4)|^q}{(2-\xi)(3-\xi)}+\frac{m}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{\mathfrak{m}_1 n}\right)\right|^q\right\}^{\frac{1}{q}}\\
&\quad+(n-2)^3\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(2-\xi)(3-\xi)}+\frac{\mathfrak{m}_1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{\mathfrak{m}_1 n}\right)\right|^q\right\}^{\frac{1}{q}}\Bigg], \quad (3.11)
\end{aligned}
$$

where $\Psi(\cdot)$ *is defined in Remark* 2.3 *and* $M(\mathfrak{T};n,\flat_3,\flat_4)$ *is the left hand side of Lemma* 3.1.

Proof. With equation (3.1) and using the $(\xi,-1,\mathfrak{m}_1)$–convexity of $|\mathfrak{T}''|^q$ and the property of modulus, we obtain

$$
\begin{aligned}
|M(\mathfrak{T};n,\flat_3,\flat_4)|\leqslant \frac{(\flat_4-\flat_3)^2}{2n^3}\Bigg[&\left\{\int_0^1\left(\mathfrak{V}-\mathfrak{V}^2\right)\,d\mathfrak{V}\right\}^{1-\frac{1}{q}}\\
\times&\left\{\int_0^1\left(\mathfrak{V}-\mathfrak{V}^2\right)\left(\mathfrak{V}^{-\xi}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^{-1}\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|^q\right)d\mathfrak{V}\right\}^{\frac{1}{q}}\\
&+\left\{\int_0^1\left(\mathfrak{V}-\mathfrak{V}^2\right)\,d\mathfrak{V}\right\}^{1-\frac{1}{q}}\\
\times&\left\{\int_0^1\left(\mathfrak{V}-\mathfrak{V}^2\right)\left(\mathfrak{V}^{-\xi}\left|\mathfrak{T}''(\flat_4)\right|^q+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^{-1}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{\mathfrak{m}_1 n}\right)\right|^q\right)d\mathfrak{V}\right\}^{\frac{1}{q}}\\
&+(n-2)^3\left\{\int_0^1\left(\mathfrak{V}-\mathfrak{V}^2\right)\,d\mathfrak{V}\right\}^{1-\frac{1}{q}}\\
\times&\left\{\int_0^1\left(\mathfrak{V}-\mathfrak{V}^2\right)\left(\mathfrak{V}^{-\xi}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{\mathfrak{m}_1 n}\right)\right|^q+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^{-1}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{\mathfrak{m}_1 n}\right)\right|^q\right)d\mathfrak{V}\right\}^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

After simplification with using Remark (2.3), we get (3.11) .
Theorem 3.9 is finished. □

Remark 3.10. If we take $n = 2$, it reduces to (Theorem 3.2, [12]).

Corollary 3.11. *Under the suppositions of Theorem 3.9, we get*

$$
\begin{aligned}
&|M(\mathfrak{T}; n, \flat_3, \flat_4)| \\
&\leqslant \frac{(\flat_4-\flat_3)^2}{2n^3} \times \left(\frac{1}{6}\right)^{1-\frac{1}{q}} \left[\left\{ \frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi-2)(\xi-3)} + \frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right) - \Psi\left(\frac{2}{\xi}\right)\right] |\mathfrak{T}''(\flat_3)|^q \right\}^{\frac{1}{q}} \right. \\
&\quad + \left\{ \frac{|\mathfrak{T}''(\flat_4)|^q}{(2-\xi)(3-\xi)} + \frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right) - \Psi\left(\frac{2}{\xi}\right)\right] \left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q \right\}^{\frac{1}{q}} \\
&\quad \left. + (n-2)^3 \left\{ \frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(2-\xi)(3-\xi)} + \frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right) - \Psi\left(\frac{2}{\xi}\right)\right] \left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q \right\}^{\frac{1}{q}} \right]. \qquad (3.12)
\end{aligned}
$$

Proof. This case is related for Theorem 3.9 by taking $m_1 = 1$. □

Theorem 3.12. *Suppose a mapping* $\mathfrak{T} : \left(0, \frac{c^*}{m_1}\right] \to \mathfrak{R}$ *is two times differentiable on* $\mathfrak{J}^\circ$ *and* $\mathfrak{T}'' \in L_1([\flat_3, \flat_4])$ *for* $\flat_3, \flat_4 \in (0, c^*]$ *with* $\flat_3 < \flat_4$. *If* $|\mathfrak{T}''|^q$ *is* (ξ, s, m_1)*–convexity on* $\left(0, \frac{c^*}{m_1}\right]$ *for* $q > 1$*and* $q \geqslant w$, *for some* $(\xi, m_1) \in (0,1]^2$ *and* $s \in (-1, 1]$, $w \geqslant 0$, *then below holds:*

$$
\begin{aligned}
|M(\mathfrak{T}; n, \flat_3, \flat_4)| &\leqslant \frac{(\flat_4-\flat_3)^2}{2n^3} \times \left(\beta\left(\frac{2q-w-1}{q-1}, \frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}} \\
&\quad \times \left[\left\{ \frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+w+1)} + m_1 \beta(w+1, \xi s+1) \left|\mathfrak{T}''\left(\frac{\flat_3}{m}\right)\right|^q \right\}^{\frac{1}{q}} \right. \\
&\quad + \left\{ \frac{|\mathfrak{T}''(\flat_4)|^q}{(\xi s+w+1)} + m_1 \beta(w+1, \xi s+1) \left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q \right\}^{\frac{1}{q}} \\
&\quad \left. + (n-2)^3 \left\{ \frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+w+1)} + m_1 \beta(w+1, \xi s+1) \left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q \right\}^{\frac{1}{q}} \right], \qquad (3.13)
\end{aligned}
$$

where $\beta(.,.)$ is defined in Remark 2.3 and $M(\mathfrak{T}; n, \flat_3, \flat_4)$ is the left hand side of equation (3.1).

Proof. With equation (3.1) and using the' integral inequality of Hölder along with (ξ, s, m_1)– convexity of $|\ \mathfrak{T}''|^q$, we have

$$
|M(\mathfrak{T}; n, \flat_3, \flat_4)| \leqslant \frac{(\flat_4-\flat_3)^2}{2n^3} \left[\left\{ \int_0^1 \mathfrak{V}^{\frac{q-w}{q-1}} (1-\mathfrak{V})^{\frac{q}{q-1}} \, d\mathfrak{V} \right\}^{1-\frac{1}{q}} \right.
$$

$$\times\left\{\int_0^1\left(\mathfrak{V}^{w}\left|\mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+(1-\mathfrak{V})\flat_3\right)\right|^{q}\right)d\mathfrak{V}\right\}^{\frac{1}{q}}$$

$$+\left\{\int_0^1\mathfrak{V}^{\frac{q-w}{q-1}}(1-\mathfrak{V})^{\frac{q}{q-1}}\,d\mathfrak{V}\right\}^{1-\frac{1}{q}}$$

$$\times\left\{\int_0^1\left(\mathfrak{V}^{w}\left|\mathfrak{T}''\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^{q}\right)d\mathfrak{V}\right\}^{\frac{1}{q}}$$

$$+(n-2)^3\left\{\int_0^1\mathfrak{V}^{\frac{q-w}{q-1}}(1-\mathfrak{V})^{\frac{q}{q-1}}\,d\mathfrak{V}\right\}^{1-\frac{1}{q}}$$

$$\times\left\{\int_0^1\left(\mathfrak{V}^{w}\left|\mathfrak{T}''\left(\mathfrak{V}\frac{\flat_3+(n-1)\flat_4}{n}+(1-\mathfrak{V})\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^{q}\right)d\mathfrak{V}\right\}^{\frac{1}{q}}\Bigg]$$

$$\leqslant\frac{(\flat_4-\flat_3)^2}{2n^3}\Bigg[\left\{\int_0^1\mathfrak{V}^{\frac{q-w}{q-1}}(1-\mathfrak{V})^{\frac{q}{q-1}}\,d\mathfrak{V}\right\}^{1-\frac{1}{q}}$$

$$\times\left\{\int_0^1\mathfrak{V}^{w}\left(\mathfrak{V}^{\xi,s}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^{q}+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^{s}\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|^{q}\right)d\mathfrak{V}\right\}^{\frac{1}{q}}$$

$$+\left\{\int_0^1\mathfrak{V}^{\frac{q-w}{q-1}}(1-\mathfrak{V})^{\frac{q}{q-1}}\,d\mathfrak{V}\right\}^{1-\frac{1}{q}}$$

$$\times\left\{\int_0^1\mathfrak{V}^{w}\left(\mathfrak{V}^{\xi,s}\left|\mathfrak{T}''(\flat_4)\right|^{q}+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^{s}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{\mathfrak{m}_1 n}\right)\right|^{q}\right)d\mathfrak{V}\right\}^{\frac{1}{q}}$$

$$+(n-2)^3\left\{\int_0^1\mathfrak{V}^{\frac{q-w}{q-1}}(1-\mathfrak{V})^{\frac{q}{q-1}}\,d\mathfrak{V}\right\}^{1-\frac{1}{q}}$$

$$\times\left\{\int_0^1\mathfrak{V}^{w}\left(\mathfrak{V}^{\xi,s}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{\mathfrak{m}_1 n}\right)\right|^{q}+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^{s}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{\mathfrak{m}_1 n}\right)\right|^{q}\right)d\mathfrak{V}\right\}^{\frac{1}{q}}\Bigg].$$

By using the calculas tools, one can have the eq. (3.13). □

Corollary 3.13. *Suppose a mapping $\mathfrak{T}:\left(0,\frac{c^*}{\mathfrak{m}_1}\right]\to\mathfrak{R}$ is two times differentiable on $\mathfrak{J}^{\circ}$ and $\mathfrak{T}''\in L_1([\flat_3,\flat_4])$ for $\flat_3,\flat_4\in(0,c^*]$ with $\flat_3<\flat_4$. If $|\mathfrak{T}''|^q$ is $(\xi,s)-$convexity on $\left(0,\frac{c^*}{\mathfrak{m}_1}\right]$ for $q>1$and $q\geqslant w$ and as for $\xi,\in(0,1]^2$ and $s\in(-1,1]$, $w\geqslant 0$, then*

$$|M(\mathfrak{T};n,\flat_3,\flat_4)| \leqslant \frac{(\flat_4-\flat_3)^2}{2n^3} \times \left(\beta\left(\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}}$$
$$\times \left[\left\{ \frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+1)} + m_1\beta\left(1,\xi s+1\right)\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|^q \right\}^{\frac{1}{q}} \right.$$
$$+ \left\{ \frac{|\mathfrak{T}''(\flat_4)|^q}{(\xi s+1)} + m_1\beta\left(1,\xi s+1\right)\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q \right\}^{\frac{1}{q}}$$
$$\left. + (n-2)^3 \left\{ \frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+1)} + m_1\beta\left(1,\xi s+1\right)\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q \right\}^{\frac{1}{q}} \right], \quad (3.14)$$

where $\beta(.,.)$ is the beta function defined in Remark 2.3.

Proof. Under the supposition in Theorem 3.12 as for $w=0$. □

Corollary 3.14. *With same assumptions in Theorem 3.12 and Corrollary (3.13), if $m_1 = 1$, then*

$$|M(\mathfrak{T};n,\flat_3,\flat_4)| \leqslant \frac{(\flat_4-\flat_3)^2}{2n^3} \times \left(\beta\left(\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}}$$
$$\times \left[\left\{ \frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+1)} + \beta\left(1,\xi s+1\right)\left|\mathfrak{T}''(\flat_3)\right|^q \right\}^{\frac{1}{q}} \right.$$
$$+ \left\{ \frac{|\mathfrak{T}''(\flat_4)|^q}{(\xi s+1)} + \beta\left(1,\xi s+1\right)\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q \right\}^{\frac{1}{q}}$$
$$\left. + (n-2)^3 \left\{ \frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+1)} + \beta\left(1,\xi s+1\right)\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q \right\}^{\frac{1}{q}} \right].$$

4. Further Results by New Hölder and Improved Power Mean Inequalities(IPM):

In [20], S. Özcan and İ. İmdat in 2019 gave a new and different representation of 'Hölder's -İşcan integral inequality' which is the modified form of 'Hölder's integral inequality'. We use the ability of this technique and obtained inequalities for the integrals which gives better results than the 'classical Hölder's integral inequality'.

Theorem 4.1. *Suppose a mapping $g : (0, \frac{c^*}{m_1}] \to \mathfrak{R}$ is two times differentiable on $\mathfrak{J}^\circ$ and $\mathfrak{T}'' \in L_1([\flat_3,\flat_4])$ for $\flat_3,\flat_4 \in (0,c^*]$ with $\flat_3 < \flat_4$. If $|\mathfrak{T}''|^q$ is (ξ,s,m_1)−convexity on $(0,\frac{c^*}{m_1}]$ for $q \geqslant 1$, as for $(\xi,m_1) \in (0,1]^2$ and $s \in (-1,1]$, then below holds:*

$$
\begin{aligned}
&|M(\mathfrak{T};n,\flat_3,\flat_4)| \\
&\leqslant N\times\left(\beta\left(p+1\;,\;p+2\right)\right)^{\frac{1}{p}}\Bigg[\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{\left(\xi s+2\right)\left(\xi s+1\right)}+\frac{m_1}{\xi s+2}\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{\left(\xi s+2\right)}+m_1\beta\left(\xi s+1\;,\;2\right)\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+\left\{\frac{\left|\mathfrak{T}''\left(\flat_4\right)\right|^q}{\left(\xi s+2\right)\left(\xi s+1\right)}+\frac{m_1}{\xi s+2}.\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+\left\{\frac{\left|\mathfrak{T}''\left(\flat_4\right)\right|^q}{\left(\xi s+2\right)}+m_1\beta\left(\xi s+1\;,\;2\right)\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+(n-2)^3\Bigg(\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{\left(\xi s+2\right)\left(\xi s+1\right)}+\frac{m_1}{\xi s+2}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{\left(\xi s+2\right)}+m_1\,\beta\left(\xi s+1\;,\;2\right)\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q\right\}^{\frac{1}{q}}\Bigg)\Bigg], \qquad (4.1)
\end{aligned}
$$

where $N=\frac{(\flat_4-\flat_3)^2}{2n^3}$ *and* $M(\mathfrak{T};n,\flat_3,\flat_4)$ *is the left hand side of Lemma 3.1.*

Proof. With equation (3.1) and using Hölder-İşcan integral inequality [Theorem 1.4, [20]] and (ξ,s,m_1)–convexity of $|\mathfrak{T}''|^q$, we have

$$
\begin{aligned}
|M(\mathfrak{T};n,\flat_3,\flat_4)| &\leqslant N\times\Bigg\{\Bigg\{\left(\int_0^1\mathfrak{V}^p\left(1-\mathfrak{V}\right)^{p+1}\,d\mathfrak{V}\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1(1-\mathfrak{V})|\mathfrak{T}''(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+(1-\mathfrak{V})\flat_3)|^q d\mathfrak{V}\right)^{\frac{1}{q}}\Bigg\} \\
+&\left\{\left(\int_0^1\mathfrak{V}^{p+1}\left(1-\mathfrak{V}\right)^{p}\,d\mathfrak{V}\right)^{\frac{1}{p}}\left(\int_0^1\mathfrak{V}\left|\mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+(1-\mathfrak{V})\flat_3\right)\right|^q d\mathfrak{V}\right)^{\frac{1}{q}}\right\} \\
+&\left\{\left(\int_0^1\mathfrak{V}^{p}\left(1-\mathfrak{V}\right)^{p+1}\,d\mathfrak{V}\right)^{\frac{1}{p}}\left(\int_0^1(1-\mathfrak{V})\left|\mathfrak{T}''\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q d\mathfrak{V}\right)^{\frac{1}{q}}\right\} \\
+&\left\{\left(\int_0^1\mathfrak{V}^{p+1}\left(1-\mathfrak{V}\right)^{p}\,d\mathfrak{V}\right)^{\frac{1}{p}}\left(\int_0^1\mathfrak{V}\left|\mathfrak{T}''\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q d\mathfrak{V}\right)^{\frac{1}{q}}\right\} \\
+&\quad(n-2)^3\times
\end{aligned}
$$

$$\left\{\left\{\left(\left(\int_0^1 (1-\mathfrak{V})\mathfrak{V}^p(1-\mathfrak{V})^p\ d\mathfrak{V}\right)^{\frac{1}{p}}\right.\right.\right.$$
$$\left.\times\left(\int_0^1 (1-\mathfrak{V})\left|\mathfrak{T}''(\mathfrak{V}\frac{\flat_3+(n-1)\flat_4}{n}+(1-\mathfrak{V})\frac{(n-1)\flat_3+\flat_4}{n})\right|^q d\mathfrak{V}\right)^{\frac{1}{q}}\right\}$$

$$+\left\{\left(\left(\int_0^1 \mathfrak{V}^{p+1}(1-\mathfrak{V})^p\ d\mathfrak{V}\right)^{\frac{1}{p}}\right.\right.$$
$$\left.\left.\left.\times\left(\int_0^1 \mathfrak{V}\left|\mathfrak{T}''\left(\mathfrak{V}\frac{\flat_3+(n-1)\flat_4}{n}+(1-\mathfrak{V})\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q d\mathfrak{V}\right)^{\frac{1}{q}}\right\}\right\}\right\}. \quad (4.2)$$

Note that

$$\int_0^1 \mathfrak{V}^p(1-\mathfrak{V})^{p+1}\ d\mathfrak{V} = \int_0^1 \mathfrak{V}^{p+1}(1-\mathfrak{V})^p\ d\mathfrak{V} = \beta(p+1\,,\ p+2). \quad (4.3)$$

Since $|\mathfrak{T}''|^q$ is $(\xi, s, \mathfrak{m}_1)-$convexity on $[\flat_3, \flat_4]$.

$$\begin{aligned}
&\int_0^1 (1-\mathfrak{V})\left|\mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+(1-\mathfrak{V})\flat_3\right)\right|^q d\mathfrak{V}\\
&\quad\leqslant \int_0^1 (1-\mathfrak{V})\left\{\mathfrak{V}^{\xi,s}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|^q\right\}d\mathfrak{V}\\
&\quad= \frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+1)(\xi s+2)}+\mathfrak{m}_1\frac{\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|^q}{\xi s+2}
\end{aligned} \quad (4.4)$$

and

$$\begin{aligned}
&\int_0^1 \mathfrak{V}\left|\mathfrak{T}''\left(\mathfrak{V}\frac{(n-1)\flat_3+\flat_4}{n}+(1-\mathfrak{V})\flat_3\right)\right|^q d\mathfrak{V}\\
&\quad\leqslant \int_0^1 \mathfrak{V}\left\{\mathfrak{V}^{\xi,s}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|^q\right\}d\mathfrak{V}\\
&\quad= \frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{\xi s+2}+\mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|^q\beta(\xi s+1\,,\ 2)
\end{aligned} \quad (4.5)$$

and

$$\begin{aligned}
&\int_0^1 (1-\mathfrak{V})\left|\mathfrak{T}''\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q d\mathfrak{V}\\
&\quad\leqslant \int_0^1 (1-\mathfrak{V})\left\{\mathfrak{V}^{\xi,s}\left|\mathfrak{T}''(\flat_4)\right|^q+\mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{\mathfrak{m}_1 n}\right)\right|^q\right\}d\mathfrak{V}\\
&\quad= \frac{|\mathfrak{T}''(\flat_4)|^q}{(\xi s+1)(\xi s+2)}+\mathfrak{m}_1\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{\mathfrak{m}_1 n}\right)\right|^q}{\xi s+2}
\end{aligned} \quad (4.6)$$

and

$$\begin{aligned}\int_0^1 \mathfrak{V}\left|\mathfrak{T}''\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q d\mathfrak{V} &\leqslant \int_0^1 \mathfrak{V}\left\{\mathfrak{V}^{\xi s}\left|\mathfrak{T}''(\flat_4)\right|^q+m_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q\right\}d\mathfrak{V}\\ &= \frac{|\mathfrak{T}''(\flat_4)|^q}{\xi s+2}+m_1\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q\beta(\xi s+1,2)\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\int_0^1 (1-\mathfrak{V})\left|\mathfrak{T}''\left(\mathfrak{V}\frac{\flat_3+(n-1)\flat_4}{n}+(1-\mathfrak{V})\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q d\mathfrak{V} &\leqslant \int_0^1 (1-\mathfrak{V})\left\{\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q+m_1.\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q\right\}d\mathfrak{V}\\ &= \frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+1)\ (\xi s+2)}+m_1\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q}{\xi s+2}\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\int_0^1 \mathfrak{V}\left|\mathfrak{T}''\left(\mathfrak{V}\frac{\flat_3+(n-1)\flat_4}{n}+(1-\mathfrak{V})\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q d\mathfrak{V} &\leqslant \int_0^1 \mathfrak{V}\left\{\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q+m_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q\right\}d\mathfrak{V}\\ &= \frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{\xi s+2}+m_1\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q\beta(\xi s+1,2).\end{aligned} \tag{4.9}$$

By connecting the equations (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), with (4.2) yields (4.1). □

Corollary 4.2. *With same assumptions as theorem* (3.5), *if we choose* $m_1 = 1$, *then below holds:*

$$\begin{aligned}|M(\mathfrak{T};n,\flat_3,\flat_4)| \leqslant N\times\left(\beta(p+1,p+2)\right)^{\frac{1}{p}}\Bigg[&\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+2)(\xi s+1)}+\frac{1}{\xi s+2}\left|\mathfrak{T}''(\flat_3)\right|^q\right\}^{\frac{1}{q}}\\ &+\left\{\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+2)}+\beta(\xi s+1,2).\left|\mathfrak{T}''(\flat_3)\right|^q\right\}^{\frac{1}{q}}\end{aligned}$$

$$+\left\{\frac{|\mathfrak{T}''(\flat_4)|^q}{(\xi s+2)(\xi s+1)}+\frac{1}{\xi s+2}\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q\right\}^{\frac{1}{q}}$$
$$+\left\{\frac{|\mathfrak{T}''(\flat_4)|^q}{(\xi s+2)}+\beta(\xi s+1\,,\,2)\left|\mathfrak{T}''\left(\frac{\flat_4+(n-1)\flat_3}{n}\right)\right|^q\right\}^{\frac{1}{q}}$$
$$+$$
$$(n-2)^3\left[\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+2)(\xi s+1)}+\frac{1}{\xi s+2}\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+2)}+\beta(\xi s+1\,,\,2)\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q\right\}^{\frac{1}{q}}\right]\Bigg],$$

where $N=\frac{(\flat_4-\flat_3)^2}{2n^3}$.

Theorem 4.3. *Suppose a mapping* $\mathfrak{T}:\left(0,\frac{c^*}{m_1}\right]\to\mathfrak{R}$ *is two times differentiable on* $\mathfrak{J}^\circ$ *and* $\mathfrak{T}''\in L_1([\flat_3,\flat_4])$ *for* $\flat_3,\flat_4\in(0,c^*]$ *with* $\flat_3<\flat_4$. *If* $|\mathfrak{T}''|^q$ *is* (ξ,s,m_1)*−convexity on* $\left(0,\frac{c^*}{m_1}\right]$, $q>1$, *for some* $(\xi,m_1)\in(0,1]^2$ *and* $s\in(-1,1]$ *, then below holds:*

$$\begin{aligned}
&|M(\mathfrak{T};n,\flat_3,\flat_4)|\\
&\leqslant N\times\left(\frac{1}{12}\right)^{\frac{1}{p}}\left\{\left(\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q\beta(\xi s+2\,,\,3)+m_1\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|^q\beta(\xi s+3\,,\,2)\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+3)\ (\xi s+4)}+\ m_1\left|\mathfrak{T}''\left(\frac{\flat_3}{m_1}\right)\right|^q\beta(\xi s+2\,,\,3)\right)^{\frac{1}{q}}\\
&\quad+\left(\left|\mathfrak{T}''(\flat_4)\right|^q\beta(\xi s+2\,,\,3)+\ m_1\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{m_1 n}\right)\right|^q\beta(\xi s+3\,,\,2)\right)^{\frac{1}{q}}\\
&\quad+(n-2)^3\left\{\left(\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q\beta(\xi s+2\,,\,3)+m_1\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q\beta(\xi s+3\,,\,2)\right)^{\frac{1}{q}}\right.\\
&\quad\left.\left.+\left(\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+3)\ (\xi s+4)}+m_1\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{m_1 n}\right)\right|^q\beta(\xi s+2\,,\,3)\right)^{\frac{1}{q}}\right\}\right\},
\end{aligned}\tag{4.10}$$

where $N=\frac{(\flat_4-\flat_3)^2}{2n^3}$ *and* $M(\mathfrak{T};n,\flat_3,\flat_4)$ *is the left hand side of Lemma 3.1.*

Proof. With equation (3.1) and applying the IPM-İşcan integral inequality [Theorem 1.5, [20]] and $(\xi, s, \mathfrak{m}_1)-$convexity of $|\ \mathfrak{T}''|^q$, we have

$$
\begin{aligned}
|M(\mathfrak{T}; n, \flat_3, \flat_4)| \leqslant N \times \Bigg\{ \Bigg\{ \left(\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\,(1-\mathfrak{V})\ d\mathfrak{V} \right)^{\frac{1}{p}} \\
\times \left(\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\,(1-\mathfrak{V}) \left| \mathfrak{T}'' \left(\mathfrak{V} \frac{(n-1)\,\flat_3 + \flat_4}{n} + (1-\mathfrak{V})\,\flat_3 \right) \right|^q d\mathfrak{V} \right)^{\frac{1}{q}} \\
+ \left(\int_0^1 \mathfrak{V}^2\ (1-\mathfrak{V})\ d\mathfrak{V} \right)^{\frac{1}{p}} \left(\int_0^1 \mathfrak{V}^2\ (1-\mathfrak{V}) \left| \mathfrak{T}'' \left(\mathfrak{V} \frac{(n-1)\,\flat_3 + \flat_4}{n} + (1-\mathfrak{V})\,\flat_3 \right) \right|^q d\mathfrak{V} \right)^{\frac{1}{q}} \Bigg\} \\
+ \Bigg\{ \left(\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\ (1-\mathfrak{V})\ d\mathfrak{V} \right)^{\frac{1}{p}} \\
\times \left(\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\ (1-\mathfrak{V}) \left| \mathfrak{T}'' \left(\mathfrak{V}\flat_4 + (1-\mathfrak{V}) \left(\frac{\flat_3 + (n-1)\,\flat_4}{n} \right) \right) \right|^q d\mathfrak{V} \right)^{\frac{1}{q}} \\
+ \left(\int_0^1 \mathfrak{V}^2\ (1-\mathfrak{V})\ d\mathfrak{V} \right)^{\frac{1}{p}} \left(\int_0^1 \mathfrak{V}^2\ (1-\mathfrak{V}) \left| \mathfrak{T}'' \left(\mathfrak{V}\flat_4 + (1-\mathfrak{V}) \left(\frac{\flat_3 + (n-1)\,\flat_4}{n} \right) \right) \right|^q d\mathfrak{V} \right)^{\frac{1}{q}} \Bigg\} \\
+ (n-2)^3 \Bigg\{ \left(\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\ (1-\mathfrak{V})\ d\mathfrak{V} \right)^{\frac{1}{p}} \\
\times \left(\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\ (1-\mathfrak{V}) \left| \mathfrak{T}'' \left(\mathfrak{V} \left(\frac{\flat_3 + (n-1)\,\flat_4}{n} \right) + (1-\mathfrak{V}) \left(\frac{(n-1)\,\flat_3 + \flat_4}{n} \right) \right) \right|^q d\mathfrak{V} \right)^{\frac{1}{q}} \\
+ \left(\int_0^1 \mathfrak{V}^2\ (1-\mathfrak{V})\ d\mathfrak{V} \right)^{\frac{1}{p}} \\
\times \left(\int_0^1 \mathfrak{V}^2\ (1-\mathfrak{V}) \left| \mathfrak{T}'' \left(\mathfrak{V} \left(\frac{\flat_3 + (n-1)\,\flat_4}{n} \right) + (1-\mathfrak{V}) \left(\frac{(n-1)\,\flat_3 + \flat_4}{n} \right) \right) \right|^q d\mathfrak{V} \right)^{\frac{1}{q}} \Bigg\} \Bigg\}.
\end{aligned}
$$

Since $|\mathfrak{T}''|^q$ is $(\xi, s, \mathfrak{m}_1)-$convexity on $[\flat_3, \flat_4]$.

$$
\begin{aligned}
&\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\ (1-\mathfrak{V}) \left| \mathfrak{T}'' \left(\mathfrak{V} \left(\frac{(n-1)\,\flat_3 + \flat_4}{n} \right) + (1-\mathfrak{V})\,\flat_3 \right) \right|^q d\mathfrak{V} \\
&\qquad \leqslant \int_0^1 \mathfrak{V}\ (1-\mathfrak{V})^2 \left\{ \mathfrak{V}^{\xi s} \left| \mathfrak{T}'' \left(\frac{(n-1)\,\flat_3 + \flat_4}{\mathfrak{m}_1 n} \right) \right|^q + \mathfrak{m}_1 \left(1-\mathfrak{V}^{\xi}\right)^s \left| \mathfrak{T}'' \left(\frac{\flat_3}{\mathfrak{m}_1} \right) \right|^q \right\} d\mathfrak{V} \\
&= \left| \mathfrak{T}'' \left(\frac{(n-1)\,\flat_3 + \flat_4}{\mathfrak{m}_1 n} \right) \right|^q \beta\,(\xi s + 2\ ,\ 3)\ + \mathfrak{m}_1 \left| \mathfrak{T}'' \left(\frac{\flat_3}{\mathfrak{m}_1} \right) \right|^q \beta\,(\xi s + 3\ ,\ 2) \qquad (4.11)
\end{aligned}
$$

and

$$\begin{aligned}
&\int_0^1 \mathfrak{V}^2\ (1-\mathfrak{V}) \left|\mathfrak{T}''\left(\mathfrak{V}\left(\frac{(\mathfrak{n}-1)\,\flat_3+\flat_4}{\mathfrak{n}}\right)+(1-\mathfrak{V})\,\flat_3\right)\right|^q d\mathfrak{V} \\
&\qquad\leqslant \int_0^1 \mathfrak{V}^2\ (1-\mathfrak{V}) \left\{\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\frac{(\mathfrak{n}-1)\,\flat_3+\flat_4}{\mathfrak{m}_1\mathfrak{n}}\right)\right|^q + \mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|^q\right\} d\mathfrak{V} \\
&\qquad\quad = \frac{\left|\mathfrak{T}''\left(\frac{(\mathfrak{n}-1)\flat_3+\flat_4}{\mathfrak{m}_1\mathfrak{n}}\right)\right|^q}{(\xi s+3)\ (\xi s+4)} + \mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{\flat_3}{\mathfrak{m}_1}\right)\right|^q \beta\left(\xi s+2\,,\,3\right)
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
&\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\ (1-\mathfrak{V}) \left|\mathfrak{T}''\left(\mathfrak{V}\flat_4+(1-\mathfrak{V})\left(\frac{\flat_3+(\mathfrak{n}-1)\,\flat_4}{\mathfrak{n}}\right)\right)\right|^q d\mathfrak{V} \\
&\qquad\leqslant \int_0^1 \mathfrak{V}\,(1-\mathfrak{V})^2 \left\{\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\flat_4\right)\right|^q + \mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{\flat_3+(\mathfrak{n}-1)\,\flat_4}{\mathfrak{m}_1\mathfrak{n}}\right)\right|^q\right\} d\mathfrak{V} \\
&\qquad = \left|\mathfrak{T}''\left(\flat_4\right)\right|^q \beta\left(\xi s+2\,,\,3\right) + \mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{\flat_3+(\mathfrak{n}-1)\,\flat_4}{\mathfrak{m}_1\mathfrak{n}}\right)\right|^q \beta\left(\xi s+3\,,\,2\right)
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
&\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\ (1-\mathfrak{V}) \left|\mathfrak{T}''\left(\mathfrak{V}\left(\frac{\flat_3+(\mathfrak{n}-1)\,\flat_4}{\mathfrak{n}}\right)+(1-\mathfrak{V})\left(\frac{(\mathfrak{n}-1)\,\flat_3+\flat_4}{\mathfrak{n}}\right)\right)\right|^q d\mathfrak{V} \\
&\qquad\leqslant \int_0^1 \mathfrak{V}\,(1-\mathfrak{V})^2 \left\{\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\frac{\flat_3+(\mathfrak{n}-1)\,\flat_4}{\mathfrak{n}}\right)\right|^q + \mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{(\mathfrak{n}-1)\,\flat_3+\flat_4}{\mathfrak{m}_1\mathfrak{n}}\right)\right|^q\right\} d\mathfrak{V} \\
&= \left|\mathfrak{T}''\left(\frac{\flat_3+(\mathfrak{n}-1)\,\flat_4}{\mathfrak{n}}\right)\right|^q \beta\left(\xi s+2\,,\,3\right) + \mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{(\mathfrak{n}-1)\,\flat_3+\flat_4}{\mathfrak{m}_1\mathfrak{n}}\right)\right|^q \beta\left(\xi s+3\,,\,2\right)
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
&\int_0^1 \mathfrak{V}^2\ (1-\mathfrak{V}) \left|\mathfrak{T}''\left(\mathfrak{V}\left(\frac{\flat_3+(\mathfrak{n}-1)\,\flat_4}{\mathfrak{n}}\right)+(1-\mathfrak{V})\left(\frac{(\mathfrak{n}-1)\,\flat_3+\flat_4}{\mathfrak{n}}\right)\right)\right|^q d\mathfrak{V} \\
&\qquad\leqslant \int_0^1 \mathfrak{V}^2\,(1-\mathfrak{V}) \left\{\mathfrak{V}^{\xi s}\left|\mathfrak{T}''\left(\frac{\flat_3+(\mathfrak{n}-1)\,\flat_4}{\mathfrak{n}}\right)\right|^q + \mathfrak{m}_1\left(1-\mathfrak{V}^{\xi}\right)^s\left|\mathfrak{T}''\left(\frac{(\mathfrak{n}-1)\,\flat_3+\flat_4}{\mathfrak{m}_1\mathfrak{n}}\right)\right|^q\right\} d\mathfrak{V} \\
&\qquad\quad = \frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(\mathfrak{n}-1)\flat_4}{\mathfrak{n}}\right)\right|^q}{(\xi s+3)\ (\xi s+4)} + \mathfrak{m}_1\left|\mathfrak{T}''\left(\frac{(\mathfrak{n}-1)\,\flat_3+\flat_4}{\mathfrak{m}_1\mathfrak{n}}\right)\right|^q \beta\left(\xi s+2\,,\,3\right)
\end{aligned} \tag{4.15}$$

and

$$\int_0^1 (1-\mathfrak{V})\,\mathfrak{V}\,(1-\mathfrak{V})\quad d\mathfrak{V} \quad=\quad \int_0^1 \mathfrak{V}^2\,(1-\mathfrak{V})\,d\mathfrak{V} \quad=\quad \frac{1}{12}. \tag{4.16}$$

By connecting the eqns. (4.11), (4.12), (4.13), (4.14), (4.15) , and (4.16) and get the eq. (4.10).

The proof is completed. □

Corollary 4.4. *With same assumptions as theorem* (3.6), *if we choose* $\mathfrak{m}_1 = 1$, *then the following inequality holds:*

$$
\begin{aligned}
|M(\mathfrak{T};n,\flat_3,\flat_4)| \leqslant N \times \left(\frac{1}{12}\right)^{\frac{1}{p}} \\
&\times \Bigg\{ \left(\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q \beta(\xi s+2\,,\,3) + \left|\mathfrak{T}''(\flat_3)\right|^q \beta(\xi s+3\,,\,2) \right)^{\frac{1}{q}} \\
&+ \left(\frac{\left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q}{(\xi s+3)\ (\xi s+4)} + \left|\mathfrak{T}''(\flat_3)\right|^q \beta(\xi s+2\,,\,3) \right)^{\frac{1}{q}} \\
&+ \left(\left|\mathfrak{T}''(\flat_4)\right|^q .\beta(\xi s+2\,,\,3) + \left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q \beta(\xi s+3\,,\,2) \right)^{\frac{1}{q}} \\
&+ (n-2)^3 \Bigg\{ \left(\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q \beta(\xi s+2\,,\,3) + \left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right|^q \beta(\xi s+3\,,\,2) \right)^{\frac{1}{q}} \\
&+ \left(\frac{\left|\mathfrak{T}''\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)\right|^q}{(\xi s+3)\ (\xi s+4)} + \left|\mathfrak{T}''\left(\frac{(n-1)\flat_3+\flat_3}{n}\right)\right|^q \beta(\xi s+2\,,\,3) \right)^{\frac{1}{q}} \Bigg\}\Bigg\}, \qquad (4.17)
\end{aligned}
$$

where $N = \frac{(\flat_4-\flat_3)^2}{2n^3}$.

5. Application to special means

Now we let us consider some special means for arbitrary different positive real numbers $\flat_3$ and $\flat_4$ (see [22]).

The arithmetic mean :
$A(\flat_3,\flat_4) = \frac{\flat_3+\flat_4}{2}$ $\quad \flat_3,\flat_4 \in R$ with $\flat_3,\flat_4 > 0$.

The geometric mean :
$G(\flat_3,\flat_4) = (\flat_3\flat_4)^{\frac{1}{2}}$ $\quad \flat_3,\flat_4 \in R$ with $\flat_3,\flat_4 > 0$.

The harmonic mean : $H(\flat_3,\flat_4) = \frac{2\flat_3\flat_4}{\flat_3+\flat_4}$ $\quad \flat_3,\flat_4 \in R\setminus\{0\}$.

The logrithemic mean :

$$
L(\flat_3,\flat_4) = \begin{cases} \flat_3 & \text{if } \flat_3=\flat_4 \text{ and } \flat_3,\flat_4>0 \\ \frac{\flat_4-\flat_3}{\ln\flat_4-\ln\flat_3} & \text{if } \flat_3\neq\flat_4 \end{cases}
$$

The Generalized logrithemic mean :

$$
L_n(\flat_3,\flat_4) = \begin{cases} \flat_3 & \text{if } \flat_3=\flat_4 \\ \frac{\flat_4^{n+1}-\flat_3^{n+1}}{(n+1)(\flat_4-\flat_3)} & \text{if } \flat_3\neq\flat_4 \end{cases}, \quad n \in Z\setminus\{-1,0\}; \flat_3,\flat_4>0
$$

Proposition 5.1. *If* $k \in Z \backslash \{-1,0\}$ *with* $\flat_3, \flat_4 > 0$ *, the we have the following inequality*

$$\left|\frac{1}{2n}\left[2A^n(\flat_3,\flat_4)+(n-1)\left\{\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)^k+\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^k\right\}\right]-L_k^k(\flat_3,\flat_4)\right|$$
$$\leqslant N\Big(\beta(p+1\,,\,p+2)\Big)^{\frac{1}{p}}k(k-1)$$
$$\times\Bigg[\left(\frac{\left|\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^{k-2}\right|^q}{(\xi s+1)\ (\xi s+2)}+\frac{\left|\flat_3^{k-2}\right|^q}{\xi s+2}\right)^{\frac{1}{q}}+\left(\frac{\left|\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^{k-2}\right|^q}{\xi s+2}+\left|\flat_3^{k-2}\right|^q\beta(\xi s+1\,,\,2)\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left|\flat_4^{k-2}\right|^q}{(\xi s+1)\ (\xi s+2)}+\frac{\left|\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)^{k-2}\right|^q}{\xi s+2}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left|\flat_4^{k-2}\right|^q}{\xi s+2}+\left|\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)^{k-2}\right|^q\beta(\xi s+1\,,\,2)\right)^{\frac{1}{q}}$$
$$+(n-2)^3\Bigg\{\left(\frac{\left|\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)^{k-2}\right|^q}{(\xi s+1)\ (\xi s+2)}+\frac{\left|\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^{k-2}\right|^q}{\xi s+2}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left|\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)^{k-2}\right|^q}{\xi s+2}+\left|\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^{k-2}\right|^q\beta(\xi s+1\,,\,2)\right)^{\frac{1}{q}}\Bigg\}\Bigg],$$

where $N=\frac{(\flat_4-\flat_3)^2}{2n^3}$.

Proof. This assertion follows from Corollary (4.2) for $\mathfrak{T}(x)=x^k$ and k is defined as above. □

Proposition 5.2. *If* $k \in Z \backslash \{-1,0\}$ *with* $\flat_3, \flat_4 > 0$ *, the we have the following inequality*

$$\left|\frac{1}{2n}\left[2A^n(\flat_3,\flat_4)+(n-1)\left\{\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)^k+\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^k\right\}\right]-L_k^k(\flat_3,\flat_4)\right|$$
$$\leqslant N\times\frac{k(k-1)}{12^{\frac{1}{p}}}\Bigg[\left(\left|\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^{k-2}\right|^q\beta(\xi s+2\,,\,3)+\left|\flat_3^{k-2}\right|^q\beta(\xi s+3\,,\,2)\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left|\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^{k-2}\right|^q}{(\xi s+3)\ (\xi s+4)}+\left|\flat_3^{k-2}\right|^q\beta(\xi s+2\,,\,3)\right)^{\frac{1}{q}}$$

$$+\left(\left|\flat_4^{k-2}\right|^q \beta(\xi s+2\,,\,3)+\left|\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)^{k-2}\right|^q \beta(\xi s+3\,,\,2)\right)^{\frac{1}{q}}$$

$$+(n-2)^3$$

$$\times\left\{\left(\left|\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)^{k-2}\right|^q \beta(\xi s+2\,,\,3)+\left|\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^{k-2}\right|^q \beta(\xi s+3\,,\,2)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left|\left(\frac{\flat_3+(n-1)\flat_4}{n}\right)^{k-2}\right|^q}{(\xi s+3)\ (\xi s+4)}+\left|\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)^{k-2}\right|^q \beta(\xi s+2\,,\,3)\right)^{\frac{1}{q}}\right\}\Bigg],$$

where $N=\frac{(\flat_4-\flat_3)^2}{2n^3}$.

Proof. This assertion follows from Corollary (4.4) for $\mathfrak{T}(x)=x^k$ and k is defined as above. □

Proposition 5.3. *Let* $p>1$ *with* $q(p-1)=p$ *and* $0<\flat_3<\flat_4$

$$\left|\left(L^{-1}(\flat_3,\flat_4)-1\right)-\frac{1}{n}A(\ln\flat_3,\ln\flat_4)-\frac{n-1}{n}A\left(\ln\left(\frac{\flat_3+(n-1)\flat_4}{n}\right),\ln\left(\frac{(n-1)\flat_3+\flat_4}{n}\right)\right)\right|$$

$$\leqslant\frac{(\flat_4-\flat_3)^2}{2n^3}\times\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\left\{\frac{\left|\frac{n^2}{((n-1)\flat_3+\flat_4)^2}\right|^q}{(\xi-2)(\xi-3)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{1}{\flat_3^2}\right|^q\right\}^{\frac{1}{q}}\right.$$

$$+\left\{\frac{\left|\frac{1}{\flat_4^2}\right|^q}{(2-\xi)(3-\xi)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{n^2}{(\flat_3+(n-1)\flat_4)^2}\right|^q\right\}^{\frac{1}{q}}$$

$$\left.+(n-2)^3\left\{\frac{\left|\frac{n^2}{(\flat_3+(n-1)\flat_4)^2}\right|^q}{(2-\xi)(3-\xi)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{n^2}{((n-1)\flat_3+\flat_4)^2}\right|^q\right\}^{\frac{1}{q}}\right].$$

Proof. This assertion follows from Corollary (3.11) for $\mathfrak{T}(x)=-\ln x$; where $x>0$. □

Remark 5.4. 1. If we choose $n=1$ in proposition(5.3), it reduces to

$$\left|\left(L^{-1}(\flat_3,\flat_4)-1\right)-A(\ln\flat_3,\ln\flat_4)\right|$$

$$\leqslant\frac{(\flat_4-\flat_3)^2}{2}\times\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[2\left\{\frac{\left|\frac{1}{\flat_4^2}\right|^q}{(\xi-2)(\xi-3)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{1}{\flat_3^2}\right|^q\right\}^{\frac{1}{q}}\right.$$

$$\left.-\left\{\frac{\left|\frac{1}{\flat_3^2}\right|^q}{(2-\xi)(3-\xi)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{1}{\flat_4^2}\right|^q\right\}^{\frac{1}{q}}\right].\quad(5.1)$$

2. If we choose $n = 2$ in proposition (5.3), it reduces to

$$\left|\left(L^{-1}(b_3,b_4)-1\right)-\frac{1}{2}A(\ln b_3,\ln b_4)-\frac{1}{2}A\left(\ln\left(\frac{b_3+b_4}{2}\right),\ln\left(\frac{b_3+b_4}{2}\right)\right)\right|$$
$$\leqslant \frac{(b_4-b_3)^2}{16}\times\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\left\{\frac{\left|\frac{4}{(b_3+b_4)^2}\right|^q}{(\xi-2)(\xi-3)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{1}{b_3^2}\right|^q\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{\left|\frac{1}{b_4^2}\right|^q}{(2-\xi)(3-\xi)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{4}{(b_3+b_4)^2}\right|^q\right\}^{\frac{1}{q}}\right]. \quad (5.2)$$

3. If we choose $n = 3$ in proposition(5.3), it reduces to

$$\left|\left(L^{-1}(b_3,b_4)-1\right)-\frac{1}{3}A(\ln b_3,\ln b_4)-\frac{2}{3}A\left(\ln\left(\frac{b_3+2b_4}{3}\right),\ln\left(\frac{2b_3+b_4}{3}\right)\right)\right|$$
$$\leqslant \frac{(b_4-b_3)^2}{54}\times\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\left\{\frac{\left|\frac{9}{(2b_3+b_4)^2}\right|^q}{(\xi-2)(\xi-3)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{1}{b_3^2}\right|^q\right\}^{\frac{1}{q}}\right.$$
$$+\left\{\frac{\left|\frac{1}{b_4^2}\right|^q}{(2-\xi)(3-\xi)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{9}{(b_3+b_4)^2}\right|^q\right\}^{\frac{1}{q}}$$
$$\left.+\left\{\frac{\left|\frac{9}{(b_3+2b_4)^2}\right|^q}{(2-\xi)(3-\xi)}+\frac{1}{\xi}\left[\Psi\left(\frac{3}{\xi}\right)-\Psi\left(\frac{2}{\xi}\right)\right]\left|\frac{9}{(2b_3+b_4)^2}\right|^q\right\}^{\frac{1}{q}}\right]. \quad (5.3)$$

6. Conclusion

In many practical investigations, it is necessary to bound one quantity by an-other. From algorithm programming point of view, it is interesting to have the classical inequalities having smallest upper limit because it plays an important role in optimization theory. We have demonstrated various new generalized integral inequalities involving (ξ, s, m_1) convex functions for extended case of s when $s \in [-1, 1]$. Consequently, we found inequalities of the Hermite Hadamrad kind. When we connect to $\Psi-$Gamma functions, the extended case for $s = -1$ is the most intriguing. We use various Hölder inequality variants, such as Hölder-İşcan inequality and Improved Power mean inequality, to analyze new upper bounds involving special functions. We hope our results attracts attentions of many researchers working in the field of inequalities and enables them to think further for other generalized convex functions. Finally, if the convex function of two variables defined in the interior and on boundary of a square, then it has a finite upper bound. This is an effective concept in optimization theory. The works above can also build-up for a convex function of two variables.

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